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PREFACE

This book is the outcome of a course given at the College of Wooster to juniors who intend either to major in physics or to proceed with the cooperative plan in engineering at the Massachusetts Institute of Technology. Prior to studying the material in this book a student is required to have had a minimum of a year's course in general college physics and the same in calculus. The course in mechanics and properties of matter lasts one semester, or half a year. Each week there are three lecture hours and a two- or three-hour period used either for discussion of problems or for laboratory work. Usually the problem sessions have taken about two-thirds and the laboratory work about one-third of the available time. All the material included in this book has never been covered in any one semester though practically all of it has been used at one time or another. Dynamics and wave motion have always been given, and then selections have been made from the remaining topics.

The underlying philosophy of this book is that physics is a unified body of knowledge and that mechanics furnishes the basic concepts for the whole of physics. Mechanics was one of the earliest of the sciences to be developed and very profoundly influenced the growth of the later sciences of sound, electricity, heat, light, and even quantum physics. The neutrino, that evasive entity associated with the emission of beta particles in radioactivity, was originally introduced in order to satisfy the fundamental laws of conservation of angular momentum and energy.

Throughout this book emphasis has been placed on the physical concepts. Although mathematics is indispensable in physics, it must nevertheless be used as the servant and not the master. In presenting some of the more difficult concepts an elementary introduction has first been given before proceeding with a more rigorous analysis. This helps a student to develop an understanding of the physical theory before becoming involved in mathematical procedures.

Vectors and some simple forms of vector analysis have been used throughout the book because of their usefulness in presenting the physical ideas in a concise manner. Since vector analysis is com-

monly used in more advanced treatises on physics, there is an advan tage in becoming acquainted with it early.

Many problems have been worked out in the text in order to aid in understanding the underlying theory. A number of carefully chosen problems are given at the end of each chapter. The more o these that a student solves for himself, the sooner he finds out wha he has not grasped and the better becomes his understanding of th subject, for "By your deeds ye shall be known."

A rather rigorous selection of topics has been made in the section on properties of matter. The subject of elasticity has been limited t a discussion of homogeneous isotropic materials, and only brie accounts of viscosity and surface tension have been given. Dimen sional analysis has been briefly introduced in order to show the impor tance of this form of analysis. A few results have been obtained b this means which could not otherwise have been handled at the leve of this book. Historical references have been made throughout th book in order to add that touch of human interest which often help a student to feel more familiar with some of the laws and principle which he is studying.

It is impossible to pay just due to the many people who, either a teachers, authors, students, or colleagues, have contributed to th point of view expressed in this book. In particular I should like t thank Drs. George Arfken and Lawrence Biedenharn, theoretica physicists at the Oak Ridge National Laboratory, for reading anc criticizing the manuscript. I should especially like to thank my wife who has carefully read and typed the manuscript

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Wooster, Ohio
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1 · KINEMATICS

1.1 Introduction

Physics, like all the sciences, has evolved from the simple to the more complex. As new observations have been made, theories have generally been modified rather than discarded. Each new theory must account for all the old phenomena as well as the new. Thus one finds the theoretical structure of physics becoming more and more complicated and correspondingly more capable of interpreting the results of a highly developed experimental structure. To understand modern science one must begin in a humble manner to master those basic ideas on which it is founded. To a large degree they are the concepts of mechanics which Galileo and Newton gave us and which were later developed in breadth and depth by other scientists. Essentially the science of mechanics gives a description of how objects change their positions with respect to time.

The science of mechanics and our modern scientific era may be considered to have started in 1638 with the publication of Galileo's book, *Two New Sciences.** This book contains some of the fundamental concepts of mechanics and even today is worthy of study. Galileo's ideas were extended by Newton, who in 1686 published *The Mathematical Principles of Natural Philosophy,* generally called *The Principia.*† In this book the laws of motion, the law of gravitation, the theory of tides and the theory of the solar system are presented. Newton shows how these laws can be used to describe the motion of the moon about the earth and the planets about the sun. At that time these laws appeared to be universal in application and provided a stimulus for further investigation which has continued to the present. Many of the results given in *The Principia* were derived by Newton from his invention of the calculus. This he accomplished independently of and about the same time as the German mathematician Leibnitz. However, this form of mathematical analysis was not used in *The Principia*

* *Two New Sciences.* Translated by Crew and de Salva, Northwestern University Press, 1932.

† *The Principia.* A revision of Motte's translation by F Cajori, University of California Press, 1934.

but was replaced by a geometrical one which makes the book somewhat difficult to read. Nevertheless every student of physics should become acquainted with this great work.

After Newton many advances were made by the French and German mathematical physicists. In 1743 D'Alembert published an important book, *Traité de Dynamique*, in which momentum and kinetic energy are clearly distinguished. Lagrange (1736–1813) published *Mécanique Analytique*, and Laplace (1749–1827) *Mécanique Céleste*. In these the concept of potential and many modern methods of analysis were first introduced. During the first half of the nineteenth century Joule performed many experiments, the interpretation of which led to the important principle of conservation of energy. The culmination of many of the advances in mechanics came with the publication in 1905 of the theory of relativity by Einstein. Although the Newtonian theory has been superseded by the Einstein theory, nevertheless for objects moving with velocities much less than the velocity of light, namely 186,000 mi per sec, the Newtonian theory is sufficiently accurate.

The discoveries in mechanics did much to further our knowledge of the sciences of electricity, sound, heat, and light. As will be shown later, there is a close analogy between the equations representing the motion of the forced oscillation of a spring, the motion of an electron in an atom acted on by electromagnetic radiation, the motion of a galvanometer coil, and the alternating current in a circuit consisting of an inductance and capacitance in series. In each of these equations both force and motion are represented, though in each the nature of the force and of the object moved is different.

The study of wave motion has likewise proceeded from the simple to the more complex. It now appears that all matter in motion exhibits wave properties. This combination of the particle and wave theories is dealt with in more advanced texts under the name of quantum mechanics. Fourier's studies in periodic heat flow gave rise to a form of mathematical analysis called the Fourier series. This analysis now has applications in all periodic phenomena, ranging from vibrating strings to radar. With this brief review of physics one can perhaps glimpse something of the unity of this science as well as the importance of mechanics in it.

At this point it might be well to comment on the role of mathematics in physics. Without mathematics the physical sciences would have been almost stillborn; they could only have been qualitative and incapable of much development. Mathematics is the tool employed in the precise analysis of physical concepts. However, without a clear

understanding of the physical concepts involved, mathematics is of little value in solving a physical problem. It is a good rule to try to express in words the physical concepts represented in any equation expressing a relationship between physical quantities.

1.2 Space and Time

A discussion of kinematics or the "geometry" of motion involves an understanding of the concepts of space and time. Many of us feel satisfied with our intuitive ideas of these concepts. We tend to feel that space and time are in no way connected with each other. That this was Newton's idea is shown by his definitions of absolute space and absolute time given at the beginning of *The Principia*. These definitions were not seriously challenged for over two hundred years. However, difficulties arose in the interpretation of certain optical phenomena which led Einstein to examine the concepts of space and time very carefully. He assumed in the theory of relativity that all common systems of coordinates are suitable for the description of physical phenomena and that the velocity of light in a vacuum is constant and independent of the motion of the source and the observer. The restricted theory applies only to those coordinate systems that move relative to each other with constant velocity. As a result of much more careful philosophical and mathematical analysis Einstein showed that space and time are interrelated. However, it is not the purpose of this book to give an account of the theory of relativity. A clear exposition is given by Einstein in his book, *The Theory of Relativity*.* It is recommended that you read this book, if possible, for it can give you much intellectual pleasure.

A comparison of the Galilean or Newtonian and Einstein transformations for an event in a system moving relative to a stationary one is given in the footnote.† From this we see that the Einstein transformation reduces to the Newtonian one for all velocities which are small

* A. Einstein, *The Theory of Relativity*, Henry Holt, New York, 1920.

† The primed system is moving relative to the unprimed one with a velocity v, as shown in Fig. 1.1. In the Einstein expression c represents the velocity of light in a vacuum, 3×10^{10} cm per sec.

The Newtonian Transformation	*The Einstein Transformation*
$x' = x - vt$	$x' = \dfrac{x - vt}{\sqrt{1 - (v^2/c^2)}}$
$t' = t$	$t' = \dfrac{t - \dfrac{vx}{c^2}}{\sqrt{1 - (v^2/c^2)}}$

compared to the velocity of light. These small velocities include all moving macroscopic objects on our earth, even the modern jet plane. It is with such relatively small velocities that the Newtonian or classical physics is concerned.

As a simple example of the kind of relativity dealt with in classical physics consider the motion of a stone dropped from a car moving along a straight road with constant speed. If air resistance is neglected, the stone appears to the driver to fall vertically down in a straight line. To a person standing beside the road the stone appears to follow a parabolic path. Which of these descriptions is correct? Both are correct if the observer's coordinate system is taken into account. In a coordinate system moving with the car the path is a straight line, whereas in a coordinate system attached to the earth the path is parabolic

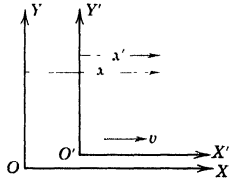


FIG. 1.1 Primed coordinate system moving relative to unprimed one

For most of our work we shall use a coordinate system attached to the earth though, as we shall see, the rotation of the earth must be taken into account in some instances. For these a coordinate system attached to the "fixed" stars is chosen. Then the motion of the earth relative to it can be taken into consideration.

1.3 Measurement of Space and Time

We are all familiar with the simple direct measurement of length in which a rod of some arbitrary length is taken as a unit and is compared with the object to be measured. In contrast to this comparatively simple method of measurement are the much more complicated and less direct methods of measuring very short and very large distances. Unfortunately there are two systems in common use which have different unit lengths as their standard. These are the British or engineering system, based on the foot, and the metric, based on the meter. To these unit lengths are added their multiples and submultiples, giving derived lengths. The relationship between the British and the metric units of length was originally

$$1 \text{ meter} = 39.37 \text{ in.}$$

but this has been slightly changed so that the relationship between the inch and the centimeter is exactly

$$1 \text{ in.} = 2.54 \text{ cm.}$$

The measurement of time is based on the rotation of the earth. One complete rotation of the earth relative to the fixed stars is an interval of time called a sidereal day and is assumed to be constant. Relative to the sun one rotation is called a true solar day. Since the sun moves eastward at a slightly varying rate with respect to the fixed stars, the true solar days are of different lengths. An average of all true solar days in a year is called a mean solar day. Our watches and clocks indicate mean solar hours, minutes, and seconds.

Since these standards of length and time may not be so constant nor so easily duplicated as is desirable for today's science, it has been recommended that atomic standards be adopted. About the beginning of the present century Michelson compared the standard meter with the wave length of the red cadmium line in air. This work has been extended so that the International Conference on Weights and Measures in 1948 endorsed the definition of the meter as equal to 1,831,249.2 waves of the green line of the mercury isotope of atomic mass number 198 whose wave length is 5460.7532×10^{-8} cm. It was also suggested at this conference that time be measured by an "atomic clock" which is based on the frequency of vibration of the ammonia molecule. Both of these units can be accurately reproduced in any suitably equipped laboratory, and both have a constancy that far exceeds our present standards. It is entirely possible that within a few years the definitions of our primary standards may be based on atomic properties.

1.4 Linear Velocity and Acceleration

In the following discussion we shall be concerned with the motion of a geometrical point or of an object whose dimensions are negligible compared to the distance necessary to fix its position. Such an object is called a particle. Thus the earth may be regarded as a particle when its motion relative to the sun is considered, but it is very far from being a particle to us moving about on its surface.

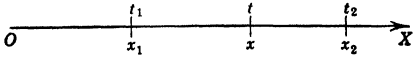


FIG. 1.2 Motion along the X axis.

Let us consider a particle moving along a straight line, which for convenience we shall take as the X axis. As the time increases, the position of the particle along the X axis changes. The origin O of the X axis, Fig. 1.2, is the reference point for locating the particle on the X axis. In agreement with the usual convention, positions to the right

of O are taken as positive and to the left of O as negative. At time t_1 the particle has the position x_1, and at time t_2 the position x_2. The distance $x_2 - x_1$ is called the displacement of the particle in the time interval $t_2 - t_1$. If, as shown in the figure, x_2 is larger than x_1, then $x_2 - x_1$ is a positive displacement whereas $x_1 - x_2$ would represent a negative displacement. The average velocity $\bar{v}$ of the particle during the time interval is defined as

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1}$$

Suppose now that we attempt to find the actual or instantaneous velocity of the particle at the instant that it is moving past position x. If the velocity of the particle is not constant, then its instantaneous velocity varies with x. To obtain the instantaneous velocity at x we find the average velocities over smaller and smaller intervals of time and then proceed to the limit where the interval approaches zero. In order to indicate small finite changes in any variable, the symbol delta Δ is used in front of the variable. Thus, if the particle is at position x at time t and after a short interval of time Δt the particle is at position $x + \Delta x$, then the average velocity $\bar{v}$ during the interval Δt is

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

The instantaneous velocity v is the limit of this expression as Δt approaches zero; that is, it is given by the derivative of x with respect to t. Thus

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

where $\dot{x}$ (x dot) is used to indicate the first derivative of x with respect to t. This is the instantaneous rate of change of the displacement x with respect to time t. If the displacement is increasing with increase in time, then $\dot{x}$ is positive; if the displacement is decreasing with increase in time, then $\dot{x}$ is negative. Notice that some convention must always be adopted for the positive direction of x.

As an example of a particle moving with non-uniform motion consider one projected vertically upward under gravity. Its displacement x from some fixed point at any time t is

$$x = ut - \tfrac{1}{2}gt^2$$

where g is a constant, x is positive in an upward direction, and at

$t = 0$ and $x = 0$ the object is moving with an upward velocity u. The instantaneous velocity of the particle at any time t is

$$\dot{x} = \frac{dx}{dt} = u - gt$$

This instantaneous velocity $\dot{x}$ decreases as the time t increases, becoming zero at a time u/g and negative thereafter. Notice that the velocity $\dot{x}$ can be negative even though the displacement is positive.

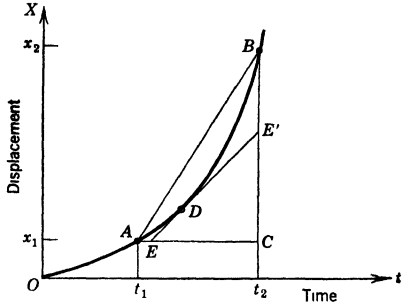


FIG. 1.3 Graphical representation of motion.

It is often helpful to represent graphically some of the concepts given above. Let the curve OAB, Fig. 1.3, represent the position x of a particle at any time t. Notice that we are considering that time can be represented by points on a straight line just as the position of a particle could be. This assumes that time varies continuously and has no sharp breaks. The position of the particle at some particular time can then be represented by a point in the plane formed by the X and t axes. At time t_1 the particle has a displacement x_1 from O and is represented by the point A in Fig. 1.3. Similarly at time t_2 the particle has a displacement x_2 from O and is represented by the point B. The average velocity of the particle from $A(x_1t_1)$ to $B(x_2t_2)$ is $(x_2 - x_1)/(t_2 - t_1)$ and is given by the slope of the chord AB or the tangent of the angle BAC. Proceeding in the manner given above, we find that the instantaneous velocity at any point D is given by the slope of the line EE' tangent to the curve at D.

When the velocity of a particle changes, the particle is said to be accelerated. By definition, *acceleration is the rate of change of velocity with time.* In the present example we are considering motion along a straight line where the acceleration consists in a change in the magnitude of the velocity and not in the direction. Later we shall discuss

accelerations where the direction changes but the magnitude of the velocity, or the speed, remains constant. Finally we shall consider accelerations where both the magnitude and the direction of the velocity of a particle change.

Suppose that a particle is moving along the X axis in such a manner that its velocity at some point x and time t is v and at $x + \Delta x$ and time $t + \Delta t$ is $v + \Delta v$. The average acceleration $\bar{a}$ in the interval of time Δt is

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

and the instantaneous acceleration a at time t is the limit of this as Δt approaches zero, or

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \dot{v}$$

where $\dot{v}$ (v dot) is the time derivative of v. Since the instantaneous velocity of this particle at position x and time t is

$$v = \frac{dx}{dt}$$

the instantaneous acceleration along the X axis may be expressed as

$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

where $\ddot{x}$ (x double dot) is the second derivative of x with respect to t. From the previous example in which $\dot{x} = u - gt$, we see that $\ddot{x}$ is a constant negative quantity, $-g$. In the upward motion of the particle the displacement and velocity are positive, or upward, whereas the acceleration is negative, or downward. For the downward motion of the particle the displacement is positive and the velocity and acceleration are both negative.

Suppose that the accceleration $\ddot{x}$ along the X axis is a constant quantity, a. By integrating a simple differential equation the velocity at any time or the distance traveled in any time may be easily obtained.

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = a$$

Integration gives

$$\int d\left(\frac{dx}{dt}\right) = \int a\, dt$$

The evaluation of this expression depends on the initial conditions. Let us assume that at time zero the particle is at the origin and is moving along the positive X axis with an instantaneous velocity u. At $t = 0$, $dx/dt = u$, and at $t = t$, $dx/dt = v$. Thus

$$\left[\frac{dx}{dt}\right]_u^v = \left[at\right]_0^t$$

or

$$v - u = at \tag{1.1}$$

This is the familiar equation for a particle moving with a constant acceleration and follows directly from the definition of acceleration. The velocity v at any time t may be given as dx/dt so that

$$\frac{dx}{dt} = u + at$$

where u and a are constants in this example. Integrating this equation and from the conditions that the displacement is zero at time $t = 0$ and that the displacement is x at time t,

$$\int_0^x dx = \int_0^t (u + at)\, dt$$

or

$$x = ut + \frac{at^2}{2} \tag{1.2}$$

The two equations 1.1 and 1.2 are two independent relationships between four of the five quantities, x, t, a, u, v. Thus three other equations may be obtained from these two by eliminating one quantity at a time. They are the ones familiar to you from your elementary physics, namely:

$$x = vt - \frac{1}{2}at^2 \qquad v^2 - u^2 = 2ax \qquad x = \frac{u + v}{2}t$$

1.5 Relative Motion

As we have already seen, all velocities and accelerations are relative to some coordinate system which we have so far considered to be at rest. There are situations in which one has to consider the motion of one body relative to another moving body. If we are sitting in a train A which is moving along tracks parallel to a second moving train B, then the motion of A relative to B depends on the velocities of both trains. Let A and B be two trains moving along the same straight

line. At time t let the position of A be x_1 and of B be x_2, Fig. 1.4. Then $x_2 - x_1$ is the displacement of B relative to A. Similarly

$$\frac{d}{dt}(x_2 - x_1) \qquad \text{and} \qquad \frac{d^2}{dt^2}(x_2 - x_1)$$

are respectively the velocity and acceleration of B relative to A. Thus

$$\frac{dx_2}{dt} - \frac{dx_1}{dt}$$

is the velocity of B relative to A and is equal to the velocity of B minus the velocity of A. If we were on train A, then we could consider ourselves at rest and train B moving relative to us with a velocity equal

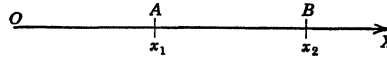


FIG. 1.4 Relative motion of B with respect to A.

to the difference of the velocities, $dx_2/dt - dx_1/dt$. Notice that we are adopting the non-relativistic point of view here, which is sufficiently accurate for all velocities that are small compared to the velocity of light.

1.6 Centripetal Acceleration

We shall now consider the acceleration of a particle the direction of whose motion is changing at a constant rate but the magnitude of whose motion is constant. A particle moving with constant speed in a circle satisfies these conditions. Let the radius of the circular path be r and the constant speed be v, Fig. 1.5. Suppose that at some time t the particle moving in the circular path is at A and after a short time Δt it is at B. In the interval of time Δt the radius OA has swept through the angle $\Delta\theta$. By definition, the angle $\Delta\theta$ in radians is the arc $AB = \Delta s$ divided by the radius:

$$\Delta\theta = \frac{\Delta s}{r} \tag{1.3}$$

In this problem the particle is moving with constant speed in a circular path and the rate of change of angle θ with time, $\Delta\theta/\Delta t$, called the *angular velocity* ω, is constant. The speed v of the particle is the rate of change of its distance with time, or

$$v = \frac{\Delta s}{\Delta t}$$

and

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\Delta s}{r\,\Delta t} = \frac{v}{r}$$

or

$$v = r\omega \tag{1.4}$$

Notice that the instantaneous angular velocity, which is the limit of $\Delta\theta/\Delta t$ as Δt approaches zero, or $d\theta/dt$, is equal to the average angular velocity $\Delta\theta/\Delta t$ since the angular velocity ω is constant.

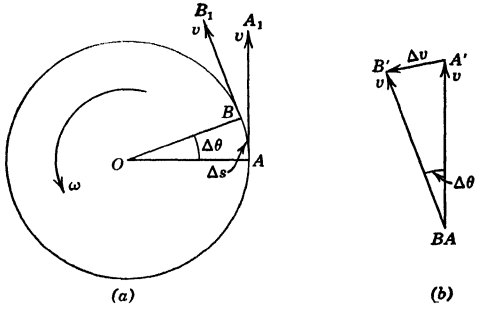


FIG. 1.5 Centripetal motion.

The instantaneous linear velocity of the particle is always tangential to the circle; at A, in Fig. 1.5*a*, it is along the tangent AA_1, and at B along the tangent BB_1. Thus the velocity is changing in direction but not in magnitude. At time t the velocity is along AA', Fig. 1.5*b*, and at time $t + \Delta t$ it is along AB' in the triangle formed by bringing the lines AA_1 and BB_1 to a common point. The lines AA' and BB' have the same length, for they are vectors representing the same magnitude of velocity.

The change in velocity during the time Δt is represented in Fig. 1.5*b* as $A'B'$ or Δv. On the same scale that AA' and BB' represent the linear velocity v, the length $A'B'$ represents the change in the velocity Δv. If $\Delta\theta$ is small, the chord $A'B'$ may be set equal to its arc and

$$A'B' = (AA')\,\Delta\theta$$

or

$$\Delta v = v\,\Delta\theta$$

so that

$$\frac{\Delta v}{\Delta t} = v\,\frac{\Delta\theta}{\Delta t}$$

This is the average acceleration and is in the direction $A'B'$. Proceed-

ing to the limit as Δt and $\Delta\theta$ approach zero, we obtain the instantaneous acceleration of the particle as

$$a = \frac{dv}{dt} = v\frac{d\theta}{dt} = v\omega$$

This acceleration is directed toward the center of the circle in which the particle is moving and is called a *centripetal* or inward acceleration. Since $v = r\omega$, the centripetal acceleration is

$$a = v\omega = r\omega^2 = \frac{v^2}{r} \tag{1.5}$$

1.7 Vectors

Quantities, such as velocity, that require both magnitude and direction to be completely described and can be added by the parallelogram law of addition are called *vector* quantities. They may be contrasted with quantities such as time, volume, speed, etc., that have magnitude only and are called *scalar* quantities. Scalar quantities expressed in the same units may be added or subtracted by the simple rules of arithmetic, but this does not hold for vector quantities. (In order to differentiate between scalar and vector quantities it is usual to print scalars in ordinary type and vectors in heavy or black type. For the blackboard or notebook it is convenient to denote a vector by placing a line below the appropriate symbol.)

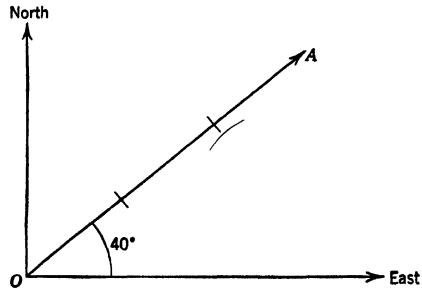


FIG. 1.6 Representation of a vector 3 units in magnitude making an angle of 40° with the east.

A vector may be represented graphically by an appropriately directed line segment whose length, drawn to a suitable scale, represents the magnitude of the vector and whose direction is indicated by an arrow drawn on the line. Thus the velocity of 30 mph, 40° N of E, may be represented by the line **OA**, Fig. 1.6, where each unit of distance

marked on **OA** represents 10 mph. Similarly the vector **OA** represents a displacement of 3 ft 40° N of E where each unit of distance on **OA** would be 1 ft.

In the discussion of centripetal acceleration we were using the law of addition of vectors when $\Delta \mathbf{v}$ was stated to represent in magnitude and direction the difference in the two velocity vectors. To understand this addition or subtraction of vector quantities let us start with displacement vectors. Thus, if a person takes 3 steps east and 4 steps northeast, the resultant of these may be found by drawing the vector

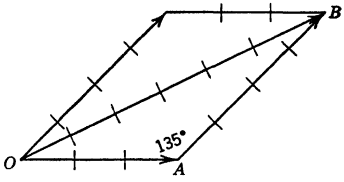


FIG. 1.7 Addition of two vectors.

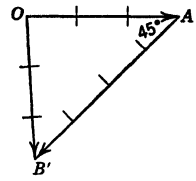


FIG. 1.8 Subtraction of two vectors by adding the negative of one of them.

diagram, Fig. 1.7, with the vectors **OA** and **AB** respectively representing in magnitude and direction the number of steps taken. The resultant displacement is represented in magnitude and direction by the vector **OB** drawn from the origin or tail of the first vector to the head of the second. The magnitude of **OB** may either be measured or calculated from the cosine formula. Thus

$$OB^2 = OA^2 + AB^2 - 2OA \times AB \cos 135°$$

$$= 9 + 16 + 24/\sqrt{2}$$

$$= 42 \quad \text{(approximately)}$$

Therefore

$$OB = 6.5 \quad \text{(approximately)}$$

The direction of the vector **OB** may be indicated by the angle BOA which **OB** makes with the easterly direction. This angle is probably most easily obtained by the sine formula.

$$\frac{OB}{\sin 135°} = \frac{BA}{\sin BOA}$$

or

$$\sin BOA = \frac{4}{6.5\sqrt{2}} = 0.435 \quad \text{(approximately)}$$

and

$$\text{angle } BOA = 25° 47' \quad \text{(approximately)}$$

The addition of the vectors has been carried out by placing the tail of the second vector at the head of the first. In this vector triangle the resultant is represented by the vector drawn from the tail of the first to the head of the second. The resultant can also be regarded as the diagonal of the parallelogram formed by placing the origin of the vectors at the same point as shown in Fig. 1.7. If more than two vectors are involved, the additional vectors are placed tail to head in a manner similar to that employed in the addition of two vectors. The resultant may be found from a scale drawing or by successive application of the cosine formula. Both methods are rather tedious, and it is more convenient to use the method of components given later.

To subtract two vectors the negative of the one to be subtracted is added to the first vector. The negative of a vector is a vector of the same magnitude but of opposite direction Thus, to subtract the two vectors in Fig. 1.7 the negative of the vector **AB,** equal to **AB′** and having a magnitude 4 and a direction SE, is added to the vector **OA,** Fig. 1.8. The resultant of the subtraction of the two vectors is given by the vector **OB′** shown in Fig. 1.8.

1.8 Components of Vectors. Unit Vectors

A vector in a plane, such as $\mathbf{OP} = \mathbf{r}$, needs two coordinates to specify its magnitude and direction. These may be its length OP and the angle θ which it makes with some specified direction, or the x and y coordinates in a Cartesian coordinate system. The vector **OP,** Fig. 1.9, can be resolved into the two vectors **OA** and **AP.** From this figure

$$\mathbf{OA} + \mathbf{AP} = \mathbf{OP}$$

The vectors **OA** and **AP** are called the components of the vector **OP.** Though it is usual to resolve a vector into components mutually at right angles, this is not a necessary condition

There is another manner of looking at a component of a vector which is often more convenient than that given above. The component of a vector **r** in the direction of a vector **p** is defined as the product of the magnitude of **r** and the cosine of the angle between the positive directions of **r** and **p.** Thus in Fig. 1.9*a* the component of the vector in the X direction is $r \cos \theta$ and in the Y direction is $r \cos \phi$. If, as here, the components are taken in directions at right angles, then $\theta + \phi = 90°$ and $r \cos \phi = r \sin \theta$.

In many problems it is convenient to introduce what is termed a

unit vector. This serves to indicate direction since its magnitude is a constant quantity, namely unity. Along the X and Y axes of a Cartesian coordinate system it is usual to denote the unit vectors by i and j respectively. If the point P, Fig. 1.9*a*, has the coordinates x and y, then the vector $\mathbf{OP} = \mathbf{r}$ is given as

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y$$

Suppose now that $\mathbf{p}_1$ is a unit vector in the direction $\mathbf{p}$, as shown in Fig. 1.9*b*, and the angle between the positive directions of vectors

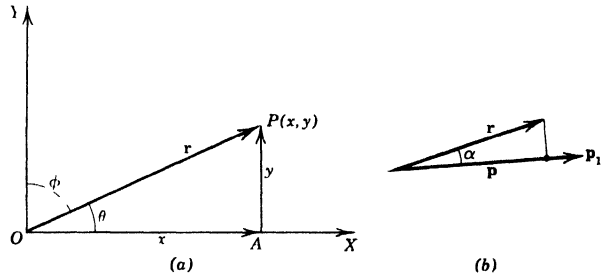


FIG. 1.9 Components of a vector.

$\mathbf{r}$ and $\mathbf{p}$ is α. The magnitude of the component of $\mathbf{r}$ in the direction of $\mathbf{p}$ is

$$r \cos \alpha$$

This magnitude is given by the dot product of the vectors $\mathbf{r}$ and $\mathbf{p}_1$, that is, by the scalar quantity

$$\mathbf{r} \cdot \mathbf{p}_1$$

By definition, the dot product of two vectors is the product of the magnitudes of the two vectors and the cosine of the angle between their positive directions. Notice that the magnitude of the unit vector $\mathbf{p}_1$ is unity. Now in order to indicate that the scalar component $\mathbf{r} \cdot \mathbf{p}_1$ is along $\mathbf{p}_1$ we write the magnitude and direction of the component as

$$\mathbf{p}_1(\mathbf{r} \cdot \mathbf{p}_1)$$

Later we shall see that two vectors can be combined to form a so-called cross product. However, no meaning is attached to the division of one vector by another vector. A vector can be multiplied or divided by a scalar, for this only affects the magnitude of the vector but not

its direction. A vector can also be differentiated with respect to a scalar but not with respect to another vector.

It is readily seen that in the addition of vectors the algebraic sum of the components in any direction is equal to the component of the

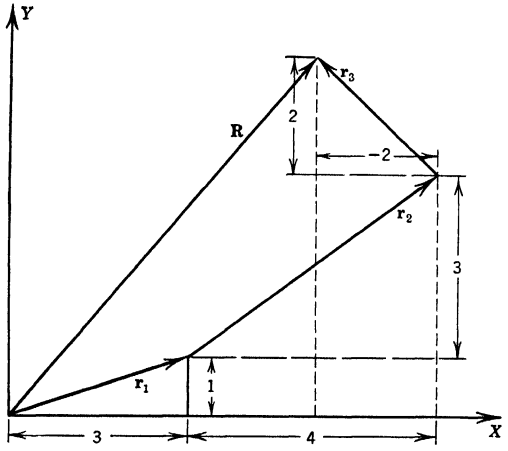


FIG. 1.10 Addition of three vectors by the polygon method.

resultant in that direction. Suppose, for example, that three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ in Fig. 1.10 are to be added, where

$$\mathbf{r}_1 = 3\mathbf{i} + \mathbf{j} \qquad \mathbf{r}_2 = 4\mathbf{i} + 3\mathbf{j} \qquad \mathbf{r}_3 = -2\mathbf{i} + 2\mathbf{j}$$

If $\mathbf{R}$ is the resultant vector, then

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$$

and if the components of $\mathbf{R}$ are $\mathbf{i}x$ and $\mathbf{j}y$, then

$$\mathbf{i}x + \mathbf{j}y = 5\mathbf{i} + 6\mathbf{j}$$

The magnitude of the X component of the resultant is

$$x = 5$$

and of the Y component is

$$y = 6$$

The magnitude of the resultant $|\mathbf{R}|$ is $\sqrt{x^2 + y^2}$ or

$$|\mathbf{R}| = \sqrt{61}$$

A space vector needs three coordinates to specify it. These may be the x, y, z coordinates in a Cartesian coordinate system, or a distance and two angles in a spherical coordinate system. In the Cartesian coordinate system $\mathbf{k}$ is used for the unit vector along the Z axis.

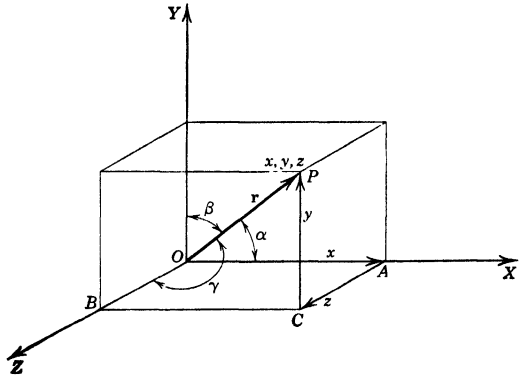


FIG. 1.11 Figure showing the space vector r with its components.

Thus the space vector $\mathbf{OP} = \mathbf{r}$, in Fig. 1.11, having coordinates x, y, z, may be written

$$\mathbf{OP} = \mathbf{OA} + \mathbf{AC} + \mathbf{CP}$$

or

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{1.6}$$

The magnitude of the vector r is

$$|\mathbf{r}| = \sqrt{(x^2 + y^2 + z^2)}$$

In general we shall employ a right-handed system of coordinate axes X, Y, Z. By this we mean that if the X axis is rotated through 90° into the Y axis this rotation will cause a right-handed screw to progress along the Z axis. Similarly rotation of the Y axis into the Z axis causes movement of a right-handed screw along the positive X axis. This rule may be easily remembered by means of Fig. 1.12 in which the X, Y, Z axes are placed in clockwise order.

If the angles that the vector r makes with the X, Y, Z axes are α, β, γ respectively, then from Fig. 1.11

$$\cos\alpha = \frac{x}{r} \qquad \cos\beta = \frac{y}{r} \qquad \cos\gamma = \frac{z}{r}$$

and

$$r^2 = x^2 + y^2 + z^2$$

Hence

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$$

or

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

The cosines of these angles are called the direction cosines of the vector **r**. Thus the sum of the squares of the direction cosines is unity.

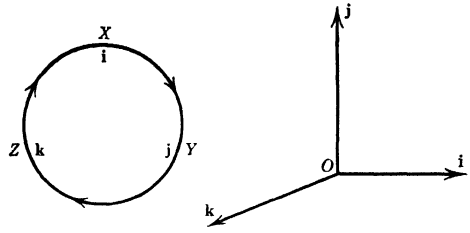


FIG 1 12 The cyclical order for the positive set of axes.

1.9 Radial and Transverse Accelerations in a Plane

We have already investigated two special cases of acceleration. In the first the magnitude but not the direction of the velocity changed; in the second the direction but not the magnitude of the velocity changed. We now turn to the general case in which a particle moves in a plane and both the magnitude and the direction of its velocity change.

Let us consider a particle moving in the XY plane, Fig. 1.13, along the path $CABD$. At every instant of time the particle has some definite position along the path. These positions may be located either with reference to a set of Cartesian coordinates X, Y or equally well with reference to polar coordinates. Thus, when the particle is at A it can be located by the x, y coordinates of this point or by the vector **r** drawn from the origin O to the point A and making an angle θ with some fixed straight line such as OX. A short time later the particle has moved to a neighboring point B, having different x, y coordinates and a different vector **r**. The displacement, velocity, and acceleration of the particle can be given in terms of the x, y coordinates or in terms of the vector **r**. Thus the components of the velocity of the particle in the X and Y directions are given respectively by the rate of

change of the x coordinate with respect to time and by the rate of change of the y coordinate with respect to time, that is, by dx/dt and dy/dt. Similarly the components of the acceleration along the X and Y directions are given respectively by d^2x/dt^2 and by d^2y/dt^2. In some problems it is convenient to express the velocity and acceleration in terms of the Cartesian coordinates; in others the polar coordinates are preferable. Since the vector representation may not be familiar to

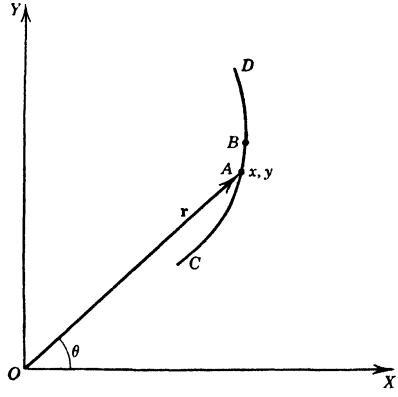


FIG. 1.13 Motion of a particle in a plane.

most of you, we shall now find the expressions for the velocity and acceleration of a particle moving in a plane in terms of vectors. The results obtained in this section can also be obtained from velocities and accelerations expressed in the Cartesian coordinate system. You are asked to do this in problems 25 and 26 at the end of this chapter.

In Fig. 1.14 let us consider a particle at a point P at a time t whose position is given by the vector $\mathbf{r}$ drawn from an arbitrary origin O and making an angle θ with some fixed direction OO'. The particle is assumed to move in the plane of the paper. It is useful to express the vector $\mathbf{r}$ in terms of its magnitude r and its direction by the unit vector $\mathbf{r}_1$, which is along $\mathbf{r}$. Thus

$$\boxed{\mathbf{r} = \mathbf{r}_1 r} \tag{1.7}$$

The instantaneous velocity of the particle is given by the time derivative of the vector $\mathbf{r}$. Thus

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}_1 \frac{dr}{dt} + r \frac{d\mathbf{r}_1}{dt} \tag{1.8}$$

In this we have applied the ordinary differentiation for a product of two quantities. The quantity $\mathbf{r}_1(dr/dt)$ is a vector that is the component of the velocity of the particle along the radius vector **OP**, or

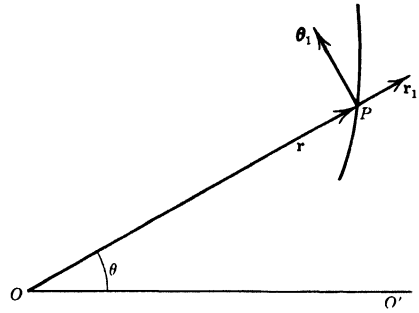


FIG. 1.14 A vector **r** showing the unit vectors $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$.

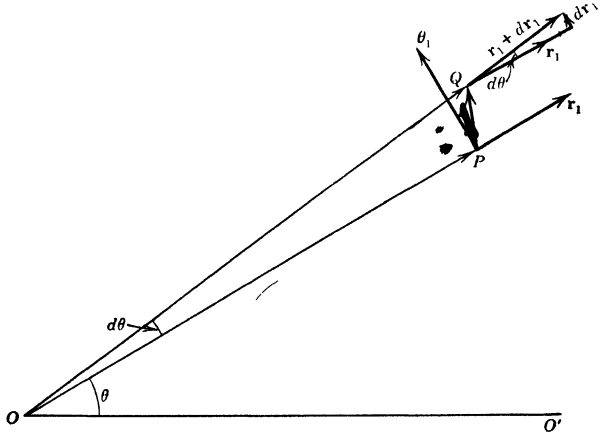


FIG. 1.15 Figure showing the vector $d\mathbf{r}_1$ equal to the vector $\boldsymbol{\theta}_1\, d\theta$.

along **r**. Now we must investigate the time derivative of the unit vector, $d\mathbf{r}_1/dt$. Since $\mathbf{r}_1$ is a unit vector it cannot change in magnitude, so that $d\mathbf{r}_1$ must be a vector in a direction at right angles to $\mathbf{r}_1$ and have no component along $\mathbf{r}_1$. This is represented in direction by the unit vector $\boldsymbol{\theta}_1$, which is at right angles to $\mathbf{r}_1$ in the direction of positive

or counterclockwise rotation. At time t the position of the particle is P, and at time $(t + dt)$ it is Q, where OP has the direction $\mathbf{r}_1$ and OQ the direction $(\mathbf{r}_1 + d\mathbf{r}_1)$. From Fig. 1.15 it is seen that $d\mathbf{r}_1$ is in the direction at right angles to $\mathbf{r}_1$ and has a magnitude of $d\theta$ since $\mathbf{r}_1$ has a magnitude of unity. Thus

$$d\mathbf{r}_1 = \boldsymbol{\theta}_1 \, d\theta$$

and

$$\frac{d\mathbf{r}_1}{dt} = \boldsymbol{\theta}_1 \frac{d\theta}{dt}$$

Hence

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}_1 \frac{dr}{dt} + \boldsymbol{\theta}_1 \frac{r \, d\theta}{dt} \tag{1.9}$$

The first term on the right-hand side of the equation is the component of velocity along the radius vector $\mathbf{r}$, and the second term is the component in a direction at right angles to this radius vector.

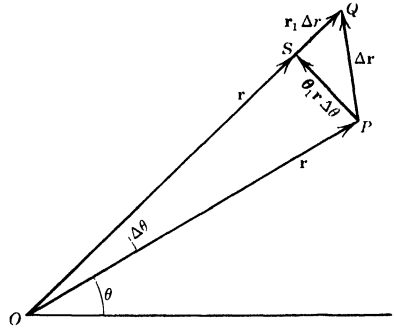


FIG. 1.16 Figure showing the vector r resolved into its two components along $\boldsymbol{\theta}_1$ and $\mathbf{r}_1$.

The physical significance of Eqn. 1.9 is brought out in Fig. 1.16. As already stated, P is the position of the particle at time t, and Q is its position at time $(t + \Delta t)$, where

$$\mathbf{OP} = \mathbf{r}$$

and

$$\mathbf{OQ} = \mathbf{r} + \Delta\mathbf{r}$$

The motion of the particle in time Δt is given in magnitude and direction by

$$\mathbf{PQ} = \Delta\mathbf{r}$$

From the vector triangle PQS, in which **PS** is at right angles to **OS**,

$$\mathbf{PQ} = \mathbf{PS} + \mathbf{SQ}$$

or

$$\Delta\mathbf{r} = \boldsymbol{\theta}_1 r\,\Delta\theta + \mathbf{r}_1\,\Delta r$$

To obtain the average velocity of the particle during the time Δt we must divide this expression by Δt, giving

$$\frac{\Delta\mathbf{r}}{\Delta t} = \boldsymbol{\theta}_1 r\frac{\Delta\theta}{\Delta t} + \mathbf{r}_1\frac{\Delta r}{\Delta t}$$

The instantaneous velocity of the particle is given by taking the limit of the above expression as Δt approaches zero, or

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\theta}_1 r\frac{d\theta}{dt} + \mathbf{r}_1\frac{dr}{dt} \tag{1.9}$$

which was obtained earlier. Notice that $d\mathbf{r}/dt$ has a magnitude of dr/dt when the motion of the particle at P is along the line OP; that is, when the particle has no angular velocity of motion about P, or $d\theta/dt = 0$.

We now turn to the acceleration of the particle, which is obtained by taking the time derivative of the velocity. From Eqn. 1.9 the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \boldsymbol{\theta}_1 r\frac{d^2\theta}{dt^2} + \boldsymbol{\theta}_1\frac{dr}{dt}\frac{d\theta}{dt} + \frac{d\boldsymbol{\theta}_1}{dt}r\frac{d\theta}{dt} + \mathbf{r}_1\frac{d^2r}{dt^2} + \frac{d\mathbf{r}_1}{dt}\frac{dr}{dt} \tag{1.10}$$

We have now to investigate the term containing $d\boldsymbol{\theta}_1/dt$, since all the other terms are known. To do this consider Fig. 1.17, in which the positions P and Q are those occupied by the particle at time t and $(t + dt)$ respectively. The vector $\boldsymbol{\theta}_1$ is perpendicular to $\mathbf{r}$, and $(\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_1)$ is perpendicular to $(\mathbf{r} + d\mathbf{r})$. Thus from the diagram it is seen that the vector $d\boldsymbol{\theta}_1$ is in the negative direction of the vector $\mathbf{r}$. The magnitude of $d\boldsymbol{\theta}_1$ is $d\theta$, so that

$$d\boldsymbol{\theta}_1 = -\mathbf{r}_1 d\theta \qquad \text{and} \qquad \frac{d\boldsymbol{\theta}_1}{dt} = -\mathbf{r}_1\frac{d\theta}{dt}$$

Substituting for $d\mathbf{r}_1/dt$ and $d\boldsymbol{\theta}_1/dt$ in Eqn. 1.10, the acceleration $\mathbf{a}$ may then be written

$$\mathbf{a} = \boldsymbol{\theta}_1 r\ddot{\theta} + \boldsymbol{\theta}_1 2\dot{r}\dot{\theta} - \mathbf{r}_1 r\dot{\theta}^2 + \mathbf{r}_1\ddot{r} \tag{1.11}$$

Collecting the terms in $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$, we have

$$\mathbf{a} = \ddot{\mathbf{r}} = \mathbf{r}_1(r - r\dot{\theta}^2) + \boldsymbol{\theta}_1(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{1.12}$$

The acceleration **a** has the components $(\ddot{r} - r\dot{\theta}^2)$ along the radius vector **r** or **OP** and $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ along the direction at right angles to *OP* in a counterclockwise or positive rotation of *OP*. We can give a physical interpretation of some of the terms in this expression for the acceleration. The component r is the outward acceleration of the particle along the radius vector, and $r\dot{\theta}^2$ is the centripetal or inward acceleration which we have previously investigated. For the trans-

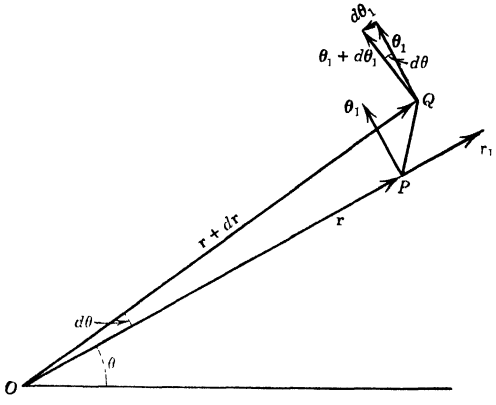


FIG. 1.17 Figure showing the vector $d\boldsymbol{\theta}_1$ equal to the vector $-\mathbf{r}_1\, d\theta$.

verse components along $\boldsymbol{\theta}_1$ the term $r\ddot{\theta}$ represents the linear acceleration of the particle at P due to the angular acceleration $\ddot{\theta}$. The last transverse component $2\dot{r}\dot{\theta}$, called the Coriolis acceleration, is not readily given a physical interpretation and probably cannot be obtained except by mathematical analysis. You can notice this acceleration if you walk radially out on a moving merry-go-round.

As a summary, the components of acceleration in a plane are as follows:

The radial component along $\mathbf{r}_1$ is $(\ddot{r} - r\dot{\theta}^2)$ (1.13)

The transverse component along $\boldsymbol{\theta}_1$ is $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ (1.14)

For a simple example in which these components are employed consider a point on the rim of a bicycle wheel which is being given an angular acceleration about the axle through its center. Let the distance of the point from the axis of rotation be a constant R. Then

$$r = R \qquad \dot{r} = 0 \qquad \ddot{r} = 0$$

At any instant of time the angular velocity is $\dot{\theta}$ and the angular acceleration $\ddot{\theta}$. Therefore the acceleration of the point on the rim is

$$\mathbf{a} = \mathbf{r}_1(-r\dot{\theta}^2) + \boldsymbol{\theta}_1(r\ddot{\theta})$$

The first term is the centripetal acceleration of the point towards the axis of rotation, and the second term is the tangential acceleration of the point due to the angular acceleration $\ddot{\theta}$ of the wheel.

1.10 Units and Dimensions of Velocity and Acceleration

Linear velocity is measured in terms of length and time. The common units of the British system are feet per second (ft per sec) and miles per hour (mph). In the metric system the units are centimeters per second (cm per sec) and meters per second. Angular velocity is always measured in radians per second in any physical equation, but this of course may be changed to degrees per second if desirable, using the relation 2π radians equal 360°.

Linear acceleration, which is defined as the time rate of change of velocity, is usually measured in the British system in feet per second per second (ft per sec^2) and in the metric system in centimeters per second per second (cm per sec^2) or in meters per second per second. Angular acceleration is expressed in radians per second per second.

To denote the dimensions of length and time the symbols L and T are adopted. In general, symbols in square brackets represent dimensions of quantities. Thus velocity has the dimensions $[L/T]$ or $[LT^{-1}]$, and linear acceleration has the dimensions $[L/T^2]$ or $[LT^{-2}]$. Since an angle in radians is defined as the arc subtended at the center of a circle divided by its radius, it follows that an angle has no dimensions. Thus angular velocity has the dimensions $[1/T]$ or $[T^{-1}]$ and angular acceleration of $[1/T^2]$ or $[T^{-2}]$.

Every term in an equation expressing a relationship between physical quantities, that is, a physical equation, must have the same units and dimensions. This basic principle of dimension theory, called the principle of dimensional homogeneity, was stated by Fourier in 1822 and by Maxwell in 1863. A discussion of the dimensions of physical concepts is given in the reference in the footnote.* For those interested in this subject many other references will be found in this article. The restriction on physical equations does not in general apply to mathematical equations. For instance, consider the linear distance s covered by a body starting from rest and moving with a linear acceleration a for a time t. The relation between these quantities is $s = at^2/2$,

* P. Moon and D. E. Spencer, "Dimensions of Physical Concepts," *Am. J. Physics*, **17**, 171 (1949).

and the velocity attained by the body in the time t is $v = at$. Both equations are dimensionally correct. Since, if equals are added to equals, the results are equal, it follows mathematically that

$$s + v = \frac{at^2}{2} + at$$

On the other hand, this equation is not dimensionally homogeneous and is of no value as a physical equation

Obviously the units employed in the various terms of a physical equation must be consistent; that is, in one term the acceleration must not be expressed in centimeters per second per second and in another in feet per second per second. It is also quite easy to make rather foolish errors in changing the units in which a given quantity is expressed. As a simple method for avoiding such errors consider changing an acceleration of 50 mph per min to feet per second per second. To do this we must change the miles to feet, the minutes to seconds, and the hours to seconds, thus.

$$\frac{50 \text{ mi}}{\text{hr} \times \text{min}} = \frac{50 \text{ mi}}{\text{hr} \times \text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{1 \text{ hr}}{3600 \text{ sec}} \times \frac{5280 \text{ ft}}{1 \text{ mi}}$$

$$= \frac{50 \times 5280 \text{ ft}}{60 \times 3600 \text{ sec} \times \text{sec}}$$

$$= 1.22 \text{ ft per sec}^2$$

Each term after the first 50 mph per min is a conversion factor whose value is unity, containing suitable units so placed that after all units are canceled out the answer is obtained in the desired units. This method of explicitly writing out all units may appear cumbersome, but it will help you to avoid many careless errors.

1.11 Setting Up and Checking of Physical Equations

Though no rules can be given for solving physical problems that will replace a sound knowledge of physical principles, the following suggestions should be of real aid.

a. After reading the problem and understanding its implications, draw a neat diagram if the situation calls for it.

b. Clearly indicate on the diagram which direction or directions you are choosing for positive, such as the positive direction for the coordinate axes, forces, velocities, accelerations, etc.

c. Set up a physical equation and explicitly state what assumptions you are making in doing this.

d. Check this equation and any derived from it for consistency of dimensions and units. It should be noted that this does not check any multiplying factors such as the ½ in the equations $s = at^2/2$.

e. After solving the equation to obtain the final answer, examine it to see if it is qualitatively correct, that is, of the right order of magnitude. This can usually be done by simplifying the original equation by omitting one or more of the less important terms and then solving the simplified equation. From the nature of the problem it should be possible to say whether the answer to the simplified equation should be greater or less than the correct answer. For example, if the time of fall of a body is to be found, taking into account the resistance of the air, this may be compared with the time of fall with no air resistance. Another method applied especially in research problems is to compare the theoretical answer with an experimental one. Though this method may not be possible with many textbook problems, it should be followed whenever practical

PROBLEMS

1. Given 1 meter = 39 37 in Find the velocity in meters per second equivalent to 60 mph. As a check on the arithmetic find the velocity, assuming the meter to be 40 in.

2. *a.* Find the acceleration due to gravity in centimeters per second per second and in meters per second per second, assuming that $g = 32$ ft per sec^2.

A body falls from rest from a height of 10,000 ft with an acceleration g

b. Find the time of fall and the velocity with which the body strikes the ground.

c. Owing to air resistance a falling body is subjected to an opposing acceleration of $(2v)$ ft per sec^2, where v is the velocity of the body at any instant of time Briefly state how the acceleration and the velocity of the body change during the motion

d. Find the constant terminal speed with which the body ultimately falls.

3. A bullet is shot vertically upwards with an initial velocity of 1200 ft per sec

a. Find the maximum elevation that the bullet attains and the total time it is in the air

b. Find the time that must elapse before the velocity of the bullet is halved, and its elevation at this time

4. The position y of a moving point is given for various times t in the table below:

t (sec)	0	1	2	3	4	5
y (cm)	2	5	6	5	2	−3

a. Plot these values with t as the abscissa or x coordinate.

b Find the average velocity of the moving point during each of the intervals 0–1, 1–2, 2–3, 3–4, 4–5 sec.

c. Find the rate of change of the average velocity during each succeeding second.

d. From the graph determine the average velocity during each of the intervals 1–2, 1–1.5, 1–1.25 sec and the instantaneous velocity at $t = 1$ sec.

e. Find the instantaneous velocity at $t = 2$, 3, 4 sec.

f. Find the rate of change of these instantaneous velocities and show that this is an acceleration of -2 cm per sec^2.

g. Assuming the acceleration to be -2 cm per sec^2 and the moving point to have an initial displacement of 2 cm and a velocity of 4 cm per sec, find the equation for the displacement at any time.

h. From this equation find the displacement at $t = 0, 1, 2, 3, 4, 5$ sec.

5. A point has a linear acceleration given by $x = 4 - 3t + 2t^2$. Initially the point has a displacement of 4 cm and a velocity of 2 cm per sec. Find the velocity and displacement of the moving point after 2 sec.

6. A moving point has a displacement given by $x = A \cos(\beta t + 1)$.

a. Show that the acceleration is $a = -\beta^2 x$.

b. Show that the velocity and displacement differ by $\pi/2$ radians at every instant of time.

c. Show that the acceleration and displacement differ by π radians at every instant of time.

7. A stone X falls vertically from rest over the edge of a cliff, and 2 sec later another stone Y is projected vertically downward with an initial velocity of 50 ft per sec. It is observed that stone Y strikes the ground 0.5 sec later than stone X. Find the height of the cliff.

8. Two sections of an express train are started 10 min apart. When starting from rest, both are capable of attaining their maximum constant speed of 60 mph in a distance of $\frac{3}{4}$ mi.

a. Show that both sections take 1.5 min to accelerate to their constant speeds.

b. Show that when the second section starts the first section is 9.25 mi ahead.

c. Show that when both trains are running at their constant speed they are 10 mi apart.

9. For a body shot along the X axis the distance x is related to the time of travel t by the equation $t = ax + bx^2$.

a. Show that the body is subjected to a retardation or negative acceleration of $2bx^3$.

b. If the body has traveled a distance of 100 ft in 0.081 sec and 200 ft in 0.164 sec, show that the value of a is 8×10^{-4} and b is 10^{-7}.

c. Show that the initial velocity of the body is 1250 ft per sec and that after 1 sec the velocity is approximately 981 ft per sec.

10. On the same track but behind a slow engine going at a constant speed of 20 mph is an express engine going 60 mph. The two engines are 1000 ft apart when the driver of the express sees the slow engine and applies the brakes. Find the minimum deceleration of the express that will just avoid a collision, i.e., so that the express attains the speed of 20 mph when it contacts the slower one.

11. A boat X is moving with a constant velocity of 4 mph, NE.

a. If boat Y is moving with a velocity of 3 mph, SW, what is its velocity relative to X? (Observer on X considers himself at rest and Y moving relative to him. Therefore bring X to rest and find the velocity of Y.)

b. Find the velocity of Y relative to X when Y is moving north with a velocity of 3 mph.

12. A motorboat having a speed of 6 mph is headed directly across a river 100 ft wide with a current of 4 mph.

a. How far down the opposite bank from the point directly opposite to the starting point does the boat land?

b. At what angle upstream must the boat be headed if it is to land at a point directly opposite its starting point?

13. A motorboat having a speed of 6 mph goes ½ mi down a river and back to its starting point. Find the difference in time taken by the boat when the river is calm and when it has a current of 2 mph.

14. A ship is moving NE by the compass through a tide that is running at 4 knots. After 2 hr the ship is 3 nautical miles SE from its starting point. Find the speed of the ship in knots and the direction of the current in the tide (*Note* 1 knot = 1 nautical mile per hour.)

15. An airplane having a cruising speed of 100 mph goes from city A to a city B, which is 400 mi NE of A. There is a wind from the south of 20 mph. Find the direction in which the plane must be headed and the time taken to go from A to B.

16. *a.* Give the dimensions of each term in the radial and transverse components of the acceleration in a plane expressed in polar components

b. What are the dimensions of the unit vectors $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$?

17. A point is whirled around in a horizontal circular path whose radius is 100 cm at a constant speed of 50 rpm.

a. Find the angular velocity in radians per second.

b. Find the centripetal acceleration.

18. The radius of the earth is approximately 4000 mi.

a. Find the angular velocity of the earth about its axis in radians per hour.

b. Find the linear velocity in miles per hour at the equator

c. Find its linear velocity at latitude 40° N.

19. Find the centripetal acceleration for a point at (*a*) the equator; (*b*) a latitude of 40° N.

20. The apparent acceleration due to gravity g is the vector sum of the earth's gravitational attraction g_0 which is directed toward the center of the earth and the centripetal acceleration toward the axis of the earth. If R is the radius and ω is the angular velocity of the earth, then show that the resultant acceleration g at any latitude θ is approximately

$$g = g_0\left(1 - \frac{\omega^2 R \cos^2 \theta}{g_0}\right)$$

(Note that ω is a relatively small quantity so that terms such as ω^4R^2/g_0^2 may be neglected as compared to 1.)

21. A circular disk with a radius of 5 cm is rotating about an axis through the center of and perpendicular to its plane with an angular acceleration of 0.5 radian per second per second. Initially at $t = 0$ the disk has an angular velocity of 2 radians per second. At the end of 10 sec find:

a. The angular velocity.

b. The angle through which the disk has turned.

c. The distance traveled by a point on the periphery.

22. A wheel whose radius is 20 cm has a constant angular acceleration of 120 rpm per sec and an initial angular velocity of 30 rpm.

a. Find the angular acceleration and the velocity in radians per second per second and radians per second respectively.

b. Find the angular velocity of a point on the periphery at the end of 0.2 sec.

c. Find the resultant linear acceleration of a point on the periphery at the end of 0.2 sec. Give both the magnitude and the direction of the resultant acceleration.

d. Find the angle through which the wheel turns in the 0.2 sec.

23. A point moves in a plane in such a way that its distance r from a fixed origin is given by

$$r = 3t + 2$$

and the angle θ which the radius vector r makes with a fixed direction by

$$\theta = 2t - t^2$$

a. Find the radial and transverse components of the acceleration of the point.

b. Find the equation of the path of the point, i.e., a relationship between r and θ not containing t.

24. A particle is placed on the spokes of a wheel in such a manner that it can slide outward from the axis. The wheel starts from rest and accelerates at the rate of 0.5 radian per second per second, with the particle increasing its radial distance r in such a manner that $r = 2t + 3t^2$. Find the resultant acceleration of the particle at the end of 2 sec.

25. Derive Eqn. 1.9, using the components of velocity x and y and the relationship from Fig. 1.13, $x = r \cos \theta$, $y = r \sin \theta$. [The component of the velocity of the particle along $\mathbf{r}_1$ is $(x \cos \theta + y \sin \theta)$. First find the component of the velocity along $\boldsymbol{\theta}_1$.]

26. Derive Eqn. 1.12, using the components of acceleration $\dot{x}$ and $\dot{y}$.

2 · FORCE AND MOTION OF PARTICLES

2.1 Introduction

In the last chapter we discussed motion and changes in motion without any regard to the manner in which these are produced. Now we turn our attention to a study of dynamics, which is concerned with forces and their effects on the motion of bodies.

Intuitively we think of changes in the motion of a body as caused by a push or a pull. A football is set in motion by a kick, a bullet by the force of expanding gases, a falling body by the pull of the earth. In each of these examples we can identify some force with the change in motion. The change in motion, or acceleration, depends not only on the size of the force but also on a property of the body moved, which Newton first called the "quantity of matter" and which later became known as the inertia or mass of the body. The difference between setting a brick and a football in motion by means of a kick will readily convince one of this property of inertia.

Furthermore the forward motion of the ball does not cease when the kick is over. The ball continues to move forward and, owing to the action of the gravitational pull of the earth, downward. The forward motion diminishes somewhat, owing to the resistance force of the air. Suppose now that the air resistance force and the gravitational pull on the earth could be eliminated. What would happen to the motion of the ball? Though it is impossible to reduce the forces on a moving body to zero, experiments show that as these forces are reduced the velocity of the body becomes more and more nearly constant. It is assumed that if the resultant force on a body were zero its velocity would remain unchanged. A force is required to change the velocity of a body, that is, to start or stop a body, to speed it up or slow it down, or to change the direction of its motion. The acceleration depends both on the force applied and on the inertia of the body. For the present we are concerned with the motion of a body as a whole and not with any rotation about an axis through the body. For such purposes it is convenient to consider the body as a particle. The quantitative measure of inertia is mass, and a simple experiment will illustrate how this may be measured.

2.2 Inertia and Mass

Suppose that two cars, A and B in Fig. 2.1, have frictionless bearings and are placed on a horizontal track. The cars are held together by a taut thread which prevents them from separating when a compressed spring is placed between them. This compressed spring produces an internal force on the system which exerts an equal and opposite force on the two cars. When the thread is burned, the cars move apart, car A moving to the right with a velocity v_A and car B to the left with a velocity v_B. The cars would then continue to move indefinitely with these velocities if the resultant force on them were zero.

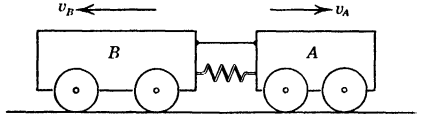


FIG. 2.1

If the changes in velocity from zero to v_A and v_B respectively take place in some short interval of time Δt, then the acceleration of car A is

$$a_A = \frac{v_A}{\Delta t}$$

and of car B is

$$a_B = \frac{v_B}{\Delta t}$$

Although Δt may not be known, it is the same for both cars. Hence

$$\frac{a_A}{a_B} = \frac{v_A}{v_B}$$

Since the velocities can be measured, it follows that the ratio of the accelerations can be determined. Suppose that the experiment is repeated with a different tension in the compressed spring. It will be found that the ratio of the accelerations is the same as that previously obtained. Further experiments would show that, no matter what the tension on the spring, the ratio of the accelerations would be constant. These accelerations depend on the inertias or masses of the cars; the larger the inertia or mass of a car, the smaller its acceleration for a given force. If m_A is the mass of car A and m_B the mass of car B, the ratio of these masses is defined as

$$\frac{m_A}{m_B} = \frac{a_B}{a_A}$$

Suppose now that car B is replaced by a car C and the experiment is repeated. Then

$$\frac{m_A}{m_C} = \frac{a_C}{a_A}$$

If car B is placed on top of car C, then

$$\frac{m_A}{m_{B+C}} = \frac{a_{B+C}}{a_A}$$

From these experiments each of the masses m_B, m_C, m_{B+C} can be obtained in terms of m_A. The results show that the total mass m_{B+C} is equal to the sum of the masses m_B and m_C. Thus mass is a scalar quantity.

Suppose that we arbitrarily assume the mass of car A to be unity. Then the inertial mass of any object can be determined by this method. Actually this dynamic method is not followed in practice, but instead some static method depending on the extension of a uniform spring or on an equal arm balance. In all these methods some arbitrary standard of mass is adopted, and then an unknown mass is determined in terms of the standard. Consistent results for the ratio of masses are obtained by any of these methods. Here the distinction between inertial and gravitational mass has been neglected. A discussion of this is given in most books on relativity.*

The unit of mass in the meter-kilogram-second or mks system is the kilogram, which is the mass of a lump of platinum-iridium carefully preserved in France. A submultiple of this, a gram or one-thousandth of a kilogram, is the unit of mass in the centimeter-gram-second or cgs system. It was originally intended that the gram should be the mass of 1 cm^3 of pure water at its temperature of maximum density, 4°C. However, more precise experiments have shown that this is not exactly correct. In the British system the unit of mass, the pound, is the mass of a lump of platinum-iridium kept in London. It turns out approximately that

$$1 \text{ lb} = 453.6 \text{ gm} \qquad \text{or} \qquad 1 \text{ kg} = 2.2 \text{ lb}$$

Another unit of mass, called the slug, has been adopted in the British engineering system of units and will be discussed later.

2.3 Force and Mass

In the discussion of inertia we stated that the velocity of a body is constant when the resultant force acting on the body is zero. This is

* A discussion of gravitational and inertial mass is given by L. T. Pockman, *Am. J. Phys.*, **19**, 305, 1951.

the basis of Newton's first law of motion, which in more formal language may be stated as follows:

Every body continues in a state of rest or in uniform motion in a straight line unless acted on by some force.

This law implies that the motion of a body is changed when an unbalanced force acts on it.

Newton's second law deals with the relationship of the force involved and the change in motion. In order to state this law Newton coined a new term, quantity of motion of a body. This is now replaced by the term momentum of a body. *Momentum* is a vector quantity having the direction of the velocity and a magnitude equal to the product of the mass and the magnitude of the velocity of the body. Just as in the discussion of velocity and acceleration we had to choose carefully the coordinate system in which the motion took place, so in using Newton's second law we must also be careful. The coordinate system to which Newton's laws are applicable is called a *primary inertial system.* A system of axes attached to the fixed stars is such a primary inertial system, and any system moving with constant velocity relative to this is an equivalent system. Although coordinates attached to the earth do not constitute an inertial system, yet for most of our problems it is sufficiently accurate to assume that they do. In a later section we shall see what effects the rotation of the earth produces. For an inertial coordinate system Newton's second law of motion may be stated as follows:

The time rate of change of momentum of a body is proportional to the resultant force acting on the body:

$$\mathbf{F} \propto \frac{d}{dt}(m\mathbf{v}) = k\frac{d(m\mathbf{v})}{dt}$$

where $\mathbf{F}$ is the resultant force acting on a body of mass m and instantaneous velocity $\mathbf{v}$. The proportionality factor k is a constant whose value can be made unity by a suitable choice of units. With such units Newton's second law may be written

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} \tag{2.1}$$

From the time of Newton to early in the present century the mass of a body was considered a constant, independent of its velocity. With the advent of relativity in 1905 Einstein showed that the mass of a body relative to an observer varies with its velocity. However, for bodies moving with velocities small compared with the velocity of light the change in mass with velocity is negligibly small, as indicated

by the equation in the footnote.* Since the velocities dealt with in classical or Newtonian physics are all negligibly small compared to the velocity of light, we shall assume that the mass of any body is a constant quantity, independent of its velocity. Thus we may write Newton's second law as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{2.2}$$

where $\mathbf{a}$ is the acceleration of a particle of mass m acted on by a resultant force $\mathbf{F}$.

Equation 2.2 deals with the vector quantities $\mathbf{F}$ and $\mathbf{a}$. Suppose that a force $\mathbf{F}$ has components F_x and F_y along the X and Y axes respectively. Then, using unit vectors as in section 1.8,

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y$$

If this force is acting on a particle of mass m, its accelerations $\ddot{x}$ and $\ddot{y}$ are given by

$$\mathbf{i}F_x = \mathbf{i}m\ddot{x} \qquad \text{and} \qquad \mathbf{j}F_y = \mathbf{j}m\ddot{y}$$

Also, if the vector $\mathbf{r}$ has the components $\mathbf{i}x$ and $\mathbf{j}y$, then

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y$$

and

$$\ddot{\mathbf{r}} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y}$$

so that

$$\mathbf{F} = m\ddot{\mathbf{r}} = \mathbf{i}m\ddot{x} + \mathbf{j}m\ddot{y}$$

Forces and accelerations may be compounded by the vector method of addition as was done above. When a particle is acted on by several forces simultaneously, the product of its mass and the resultant acceleration is equal to the resultant of the forces compounded vectorially. When the force and acceleration are resolved in any direction, the product of mass and acceleration in that direction is equal to the component of force in the same direction.

* According to the theory of relativity the mass m of a body moving with velocity v relative to a coordinate system in which its mass is m_0 is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where c is the velocity of light in a vacuum.

2.4 Kinetic Reaction. Centrifugal Force

Newton's second law may also be written $\mathbf{F} - m\mathbf{a} = 0$. The vector quantity $-m\mathbf{a}$ is called the *kinetic reaction*, or sometimes the inertia reaction. In his *Traité de Dynamique*, published in 1743, D'Alembert enunciated a principle now known as D'Alembert's principle, which may be stated as follows: When any particle is acted on by an external force, the resultant of this external force and the kinetic reaction of the particle is zero. Familiar examples of kinetic reaction are observed when we are thrown backward by a streetcar's starting suddenly, or forward by a car's stopping suddenly, or outward by a car's rounding a corner at high speed. In the last example it is usual to call this kinetic reaction the centrifugal force.

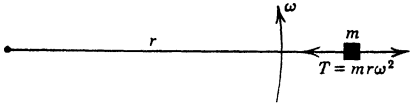


FIG 2 2 Centripetal and centrifugal forces.

Suppose that a particle of mass m is whirled around on a cord of length r with an angular velocity ω, Fig. 2.2. From Eqn. 1.5 the centripetal acceleration of the particle is $r\omega^2$. Hence the tension in the string, i.e., the centripetal force, is $mr\omega^2$, which is the inward pull of the string on the mass m. The force that the moving mass m exerts on the string or on the hand holding it is the kinetic reaction $mr\omega^2$ acting outward. This force, which does not act on mass m but on the means of constraint, is called the centrifugal force. Thus we may consider the moving mass m to be in equilibrium under the action of the force T inward and the kinetic reaction outward. Essentially the application of D'Alembert's principle changes a problem of dynamics into one of statics.

2.5 Units of Force

The manner in which the force and mass units are defined differs in the metric and the British engineering systems. In the metric system it is usual to take mass as the fundamental quantity and force as a derived quantity. The unit of force in the mks system is called the *newton*. It is that force which when acting on a mass of 1 kg gives it an acceleration of 1 meter per second per second. Similarly the unit of force in the cgs system is called the *dyne* and is that force which when acting on a mass of 1 gm gives it an acceleration of 1 cm per sec^2.

From these definitions it follows that

$$1 \text{ newton} = 10^5 \text{ dynes}$$

If the unit of mass in the British system is taken as 1 lb, then the unit of force in this system is called the poundal and is that force which when acting on a mass of 1 lb gives it an acceleration of 1 ft per sec^2. The poundal is not commonly employed as a unit of force

The above units are sometimes called the absolute system of units, although there is really nothing absolute about them They refer to a system in which mass is taken as fundamental and force is a derived quantity. For such a system the dimensions of force are given in terms of the dimensions of mass $[M]$, length $[L]$, and time $[T]$ from Newton's second law as

$$[F] = [MLT^{-2}]$$

There is also the gravitational system of units which we shall discuss in the next section

2.6 Mass and Weight

We have previously stated that at any particular location on the earth all bodies fall vertically downward with the same acceleration if air resistance is absent. Newton demonstrated this by the so-called feather and guinea experiment in which a feather and a small coin were allowed to fall in an evacuated tube It was found that the light and the heavy body fell with the same acceleration. This acceleration, called the acceleration due to gravity and denoted by g, is not exactly the same at different elevations and places on the earth. For sea-level elevation the value of g is about 983.21 cm per sec^2 at the poles and decreases to about 977.99 cm per sec^2 at the equator of the earth In the middle latitudes the value of g may be taken as approximately 980 cm per sec^2 or 32 ft per sec^2.

Since a body of mass m falls with an acceleration $\mathbf{g}$, it follows that there is a vertically downward force of $m\mathbf{g}$ on this body This force due to the earth's gravity is called the weight $\mathbf{w}$ of the body. Thus

$$\mathbf{w} = m\mathbf{g}$$

A body whose mass is 1 gm has a weight of 1 gram weight (gmw) and exerts a force of approximately 980 dynes:

$$1 \text{ gmw} = 980 \text{ dynes}$$

$$1 \text{ kgw} = 9.8 \text{ newtons}$$

In the British absolute system it similarly follows that 1 pound weight

(lbw), the gravitational force on 1 pound mass, is approximately equal to 32 poundals.

It is common practice in engineering to use the British engineering or gravitational system of units. In this system force is taken as a fundamental quantity and the unit of mass is derived. The unit of force in the engineering system is taken as the gravitational pull on a pound mass at sea level and 45° N latitude; that is, the unit of force is 1 lbw. For this unit of force, what unit of mass has to be used in Newton's second law? This unit of mass is called a *slug*. It is that mass which, when acted on by a force of 1 lbw, acquires an acceleration of 1 ft per sec^2. Since a force of 1 lbw gives a mass of 1 lb an acceleration of 32 ft per sec^2, it follows that the slug is 32 times as large as the pound mass

Writing $[F]$ for the dimension of force, it follows that in the gravitational system mass $[M]$ has the dimensions $[FT^2L^{-1}]$. In this system Newton's second law is written

$$\mathbf{F} = \frac{w}{g}\mathbf{a}$$

The forces F and w are in pounds weight, and the accelerations a and g are in feet per second per second. In general we shall use the metric system, but most of the equations may be written immediately in the engineering system by replacing the mass m by w/g.

2.7 Newton's Third Law. Conservation of Momentum

You have probably noticed that forces always occur in pairs although acting on different objects. For example, when a gun is fired there is a forward force on the bullet and an equal but opposite force on the gun. This force is sometimes called the "back-kick" of the gun. In walking, one pushes back on the earth and the earth exerts a forward force on the person. If there is no friction, as on perfectly smooth ice, one cannot push backwards. There is then no forward force, and one cannot make any headway. A propeller pushes back on the air, and the air pushes the propeller and airplane forward. When we observe any one force, it is an interesting game to pick out the other force acting on some other body. The principle involved in these and other examples is summarized in *Newton's third law of motion*, which may be stated as follows:

To every force acting on one body there is an equal and opposite force acting on another body

If $\mathbf{F}_{12}$ is the force exerted by body 2 on body 1 and $\mathbf{F}_{21}$ is the force

exerted by body 1 on body 2, then Newton's law may be mathematically stated:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

where the negative sign indicates that the forces are in opposite directions.

Let us now consider the interaction of two bodies. The forces of interaction may be due to spring connections, to electrical charges, to collisions, etc. If the forces of interaction are the only forces acting on the bodies, the system is a closed or isolated one. From Newton's second and third laws the important principle of conservation of momentum can be derived for an isolated system.

Consider an isolated system consisting of two particles m_1, m_2 moving with velocities $\dot{\mathbf{x}}_1$, $\dot{\mathbf{x}}_2$ respectively in which the forces of interaction are $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$. Then by Newton's third law

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

and by Newton's second law

$$\frac{d}{dt}(m_1\dot{\mathbf{x}}_1) = -\frac{d}{dt}(m_2\dot{\mathbf{x}}_2)$$

or

$$\frac{d}{dt}(m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2) = 0$$

But $m_1\dot{\mathbf{x}}_1$ is the momentum of particle 1, and $m_2\dot{\mathbf{x}}_2$ that of particle 2, so that the rate of change of the total momentum is zero and independent of time. This is the principle known as the conservation of momentum. It may be stated as follows:

If no external forces act on a system, i.e., if the system is isolated, the total momentum in any direction remains constant.

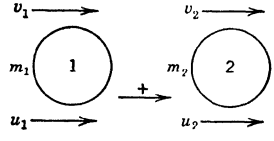


FIG. 2.3 Collision of two spheres and conservation of momentum.

As an example of the interaction between two bodies consider the impact of two spheres moving along the same straight line. Let the spheres have masses m_1, m_2 and velocities u_1, u_2 before impact and v_1, v_2 after impact, as shown in Fig. 2.3. If the positive direction of motion is to the right, then all the velocities shown are positive. In order for there to be an impact u_1 must be larger than u_2, and if the spheres are to separate v_2 must be larger than v_1. The forces at impact on the two masses are

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

By Newton's second law

$$\frac{m_1(v_1 - u_1)}{\Delta t} = -\frac{m_2(v_2 - u_2)}{\Delta t}$$

where Δt is the short time interval of impact during which the velocity of m_1 changes from u_1 to v_1 and of m_2 changes from u_2 to v_2. Transposing and canceling, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{2.3}$$

The left-hand side of the equation represents the total momentum of the two balls before collision, and similarly the right-hand side the total momentum after the collision. Thus there is no momentum lost in the collision; that is, the momentum is conserved at the collision. One cannot overemphasize the importance of the law of *conservation of momentum* in science. It is applied universally in physics whether such large objects as stars or such submicroscopic objects as atoms are dealt with.

Momentum is a vector quantity and has the same direction as the velocity of the object. For a shell exploding in the air the momentum of the shell immediately before the explosion is equal to the total momentum of the system just after the explosion. Fragments of the shell are presumably sent in every direction. In any direction the total momentum of the fragments after the explosion is equal to the momentum of the shell in the given direction before the explosion. If just before the explosion the shell is moving downward, then after the explosion the total momentum of the fragments added vectorially would be downward and the total momentum in the horizontal direction would be zero.

Before proceeding with an example of the components of momentum, it is worth while noting that Eqn. 2.3 does not give the velocities v_1, v_2 of the two masses, since a single equation can give the value of only one unknown. Further information concerning the impact is necessary if these velocities are to be found. This information is given by what Newton termed the *coefficient of restitution* e, which is defined as the ratio of the relative velocity of separation to the relative velocity of approach. For the two impacting balls shown in Fig. 2.3 the coefficient of restitution e is

$$e = \frac{v_2 - v_1}{u_1 - u_2} \tag{2.4}$$

There are two limiting values for e, namely $e = 0$ and $e = 1$. An impact for which $e = 0$ is called an *inelastic impact*, and $v_2 = v_1$ or the

bodies stick together after impact. The other case, $e = 1$, is an ideal one never actually attained in practice. For this bodies would move apart with the same relative velocity as that of approach, and the impact would be *perfectly elastic.*

As an example of impact consider two smooth billiard balls A and B. Suppose that ball A has a mass of 100 gm and on impact an initial velocity of 20 cm per sec at an angle of 30° with the line of centers. Similarly ball B has a mass of 50 gm

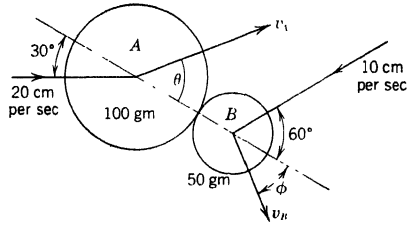


FIG. 2.4 Collision of two smooth spheres.

and a velocity of 10 cm per sec at an angle of 60° with the line of centers. The velocities after impact are given in Fig. 2.4. With a coefficient of restitution of 0.9 between the balls the problem is to find the velocities after the impact.

Since the spheres are assumed to be smooth, there is no force perpendicular to the line of centers. Thus the component of the momentum of either ball does not change along this direction. Hence, resolving the velocities or momenta along the line perpendicular to the line of centers of impact, we have for ball A

$$20 \sin 30° = v_A \sin \theta \qquad \text{or} \qquad v_A \sin \theta = 10 \text{ cm per sec}$$

and for ball B

$$10 \sin 60° = v_B \sin \phi \qquad \text{or} \qquad v_B \sin \phi = 5\sqrt{3} \text{ cm per sec}$$

By the principle of conservation of momentum there is no change at impact in the total momentum of the two balls in any direction The most convenient direction to choose is that along the line of centers, and from the law of conservation of momentum it follows that

$$2000 \cos 30° - 500 \cos 60° = 100 v_A \cos \theta + 50 v_B \cos \phi$$

or

$$2v_A \cos \theta + v_B \cos \phi = 20\sqrt{3} - 5 = 29.64$$

From the definition of the coefficient of restitution it follows that

$$0.9 = \frac{v_B \cos \phi - v_A \cos \theta}{20 \cos 30° + 10 \cos 60°}$$

or

$$v_B \cos \phi - v_A \cos \theta = 9\sqrt{3} + 4.5 = 20.09$$

There are now four unknowns and four independent equations. Solving these equations gives

$$\tan \theta = 3.142 \qquad \theta = 72° 21' \qquad v_A = 10.48 \text{ cm per sec}$$

$$\tan \phi = 0.372 \qquad \phi = 20° 21' \qquad v_B = 21.80 \text{ cm per sec}$$

Hence as a result of the collision ball A and ball B move approximately in the directions shown in Fig. 2.4.

2.8 Momentum Change and Impulse

In practice there are many examples of a large force acting for a very short time to produce a momentum change in some object. A few familiar examples are a hammer striking a blow, a bullet propelled from a gun, and a charge of electricity setting the coil of a ballistic galvanometer in motion. Even though the variation of the force with

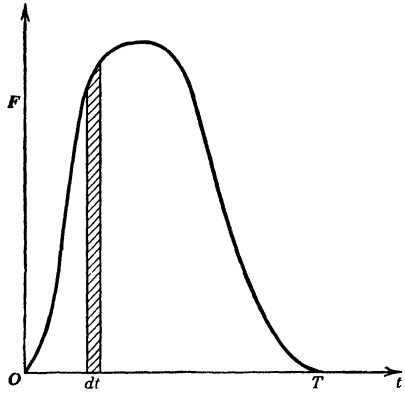


FIG 2.5 Impulse represented as the area under a force-time curve.

time may not be known, the force-time integral is a definite quantity. If the variation of force with time is that shown in Fig. 2.5, the integral $\int F\,dt$ is given by the area under the curve. This integral is called the *impulse* of the force. Suppose that at any time t a force $\mathbf{F}$ is acting on a particle mass m and velocity $\mathbf{v}$. By Newton's second law

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$$

By integration

$$\int_0^T \mathbf{F}\,dt = \int_{\mathbf{v}_0}^{\mathbf{v}} d(m\mathbf{v})$$

where $\mathbf{v}_0$ is the velocity at the time $t = 0$ when the force starts to act, and $\mathbf{v}$ is the velocity at time $t = T$ when the force is again zero. Thus the impulse is

$$\int_0^T \mathbf{F}\, dt = m\mathbf{v} - m\mathbf{v}_0 \tag{2.5}$$

The impulse of a force acting on a particle is equal to the momentum change of the particle Impulse is a vector quantity, as is momentum It follows that the component of the impulse in any direction is equal to the momentum change in that direction. When applying the impulse-momentum relationship one must be careful to maintain a consistent set of units, namely, those of Newton's second law.

2.9 Examples Involving Newton's Second Law

Our first problem is to determine the acceleration of two unequal masses hanging on the ends of a cord passing over a pulley, Fig. 2.6. Such a system is known as an Atwood machine after George Atwood, a professor at Cambridge University about 1780. For simplicity we shall make the following assumptions about this apparatus.

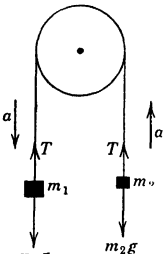


FIG. 2.6 Atwood's machine.

1. The cord is weightless.
2. The cord is perfectly flexible and inextensible.
3. The pulley is weightless and smooth.

Obviously these assumptions are impossible to attain in practice, but they may be so approximated that there is only a small error introduced.

Let us assume that the mass m_1 is larger than m_2. Then m_1 has a downward acceleration and m_2 an equal upward acceleration, since the string is inextensible. The forces T marked on the cords represent the forces exerted by the cords on the masses m_1, m_2. It requires all three assumptions to justify considering the tensions equal.

Consider first the mass m_1 on which there is a resultant force $(m_1g - T)$ acting downward. The downward acceleration a of mass m_1 is given by Newton's second law as

$$(m_1g - T) = m_1a$$

Similarly for m_2 with its upward acceleration a and upward resultant force $(T - m_2g)$ we have

$$T - m_2g = m_2a$$

Adding these two equations gives

$$a = \frac{(m_1 - m_2)g}{(m_1 + m_2)}$$

This equation could have been written down immediately for the system as a whole, since the resultant force is $(m_1 - m_2)g$ and the total mass moved is $(m_1 + m_2)$. By eliminating the acceleration a, the tension T in the cord is

$$T = \frac{2m_1m_2g}{(m_1 + m_2)}$$

This tension must have a value between m_1g and m_2g and be expressed in units of either dynes or newtons if the metric system is employed.

In the British engineering system the above equations would be written

$$W_1 - T = \frac{W_1a}{g} \qquad T - W_2 = \frac{W_2a}{g}$$

$$a = \frac{(W_1 - W_2)g}{W_1 + W_2} \quad \text{and} \quad T = \frac{2W_1W_2}{(W_1 + W_2)}$$

In this example T is in pounds weight since W is in pounds weight

As a modification of Atwood's machine let us consider the system shown in Fig. 2.7. All assumptions given above apply here except that we shall now consider the movable pulley to have a mass m but to have a negligible rotational inertia. Suppose the mass m_3 to be sufficiently large to move downward with an acceleration a'. Then the movable pulley of mass m moves upward with the acceleration a'. If m_1 is larger than m_2, then, relative to the center of the moving pulley, m_1 moves downward with an acceleration a and m_2 moves upward with the same acceleration

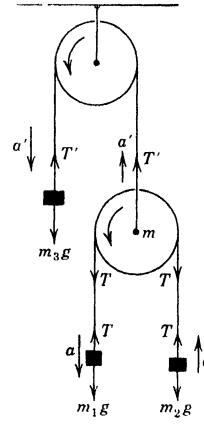


FIG. 2.7 Linear accelerations in a modified Atwood machine.

Since the center of the moving pulley has an upward acceleration a', the acceleration of m_1 relative to a stationary object is $(a - a')$ downward and that of m_2 is similarly $(a + a')$ upward. By Newton's second law the equations of motion are:

$$\begin{aligned}
&\text{For } m_3\text{:} && (m_3g - T') && = m_3a'\\
&\text{For } m_1\text{.} && (m_1g - T) && = m_1(a - a')\\
&\text{For } m_2\text{:} && (T - m_2g) && = m_2(a + a')\\
&\text{For } m\text{:} && (T' - 2T - mg) && = ma'
\end{aligned}$$

If the movable pulley had zero mass, $m = 0$, T' would be equal to $2T$. Since this is not so, there must be a resultant force acting on m to give it the upward acceleration a' It can be seen that in this problem a carefully drawn diagram with the forces and accelerations correctly placed is almost indispensable.

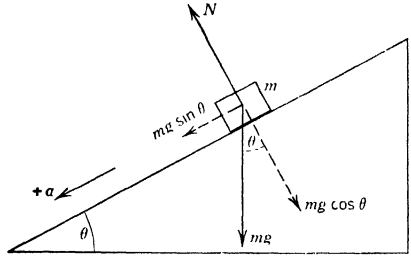


FIG. 2.8 Acceleration on a smooth inclined plane

As another application of Newton's second law let us determine the acceleration of a body on a smooth inclined plane The assumption that a plane is smooth means that there is no friction force parallel to the plane opposing the motion of the mass. Since the mass moves down the plane, its acceleration a is also in this direction. The resultant force along the plane is the component of the weight mg in this direction. By resolving the vertical force mg along and at right angles to the plane we find that the component along the plane is $mg \sin \theta$ and the component at right angles to the plane is $mg \cos \theta$, as shown in Fig. 2.8. This latter component is balanced by the force N, called the normal reaction exerted by the plane on the mass m. Since the mass m has no motion at right angles to the plane, it follows from Newton's second law that

$$mg \cos \theta - N = 0$$

The equation of motion of the mass m down the plane is

$$mg \sin \theta = ma$$

or

$$a = g \sin \theta$$

In practice it is impossible to have a perfectly smooth plane. There is always a frictional force present, and this we shall now discuss.

2.10 The Force of Friction

In general, when any two objects are in contact and the one is moved with respect to the other, there is a force at the plane of contact opposing the motion. This force is called the force of friction. Though the force of friction has some annoying aspects, it also has many useful ones. Without friction we could not walk, automobile wheels would spin around without producing any forward motion, and power could not easily be transmitted from a motor to a machine. On the other hand, friction produces power losses in moving machinery. Mechanical energy is always transformed into heat by the work done against the force of friction.

It appears that Leonardo da Vinci (1452–1519), with amazing practical genius and insight, was the first to present the laws of friction. They may be stated in the following manner: The force of friction is (*a*) independent of the area of the sliding surfaces and (*b*) proportional to the load between them. Apparently these laws were not commonly known, for in 1699 a French engineer, Amontons, independently discovered them. About 1781 Coulomb verified Amontons' observations and also made a clear distinction between the force of static friction, that is, the force required to start sliding, and kinetic or sliding friction, that is, the force required to maintain sliding. Coulomb showed that sliding friction is appreciably smaller than static friction and also that sliding friction is approximately independent of the speed of sliding.

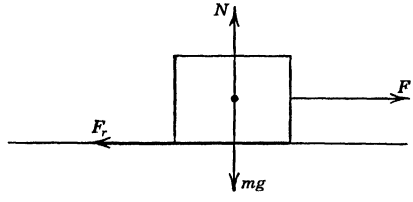


FIG. 2.9 Friction on a horizontal plane

In order to understand some of the characteristics of the force of friction let us consider a block placed on a horizontal surface, as in Fig. 2.9. A horizontal cord is attached to the block, and the tension in it is increased until it reaches some value F_r at which sliding commences. If the force is smaller than F_r, no motion results since the force of friction is a self-adjusting force opposing any motion. When sliding

starts without any appreciable acceleration, the frictional force F_r is equal to the applied force F. The normal reaction N exerted by the surface on the block is equal to the weight of the block in this example. The results of experiments show that for sliding there is an approximately constant ratio between the force of friction F and the normal reaction N. This ratio is called the coefficient of friction and is denoted by the symbol μ.

$$\mu = \frac{F}{N} \qquad \text{or} \qquad F = \mu N \tag{2.6}$$

If a weight is placed on the upper surface of the block, both N and F are increased in the same ratio so that F/N or μ continues to have the same value.

The values of the coefficient of friction between the various surfaces given in the table below are only approximate since a slight change in the composition of the surface can make a fairly large change in the coefficient.

APPROXIMATE VALUES OF THE COEFFICIENT OF FRICTION

Substances	Coefficient of Friction μ
Wood on wood, dry	0 25–0 50
Wood on wood, soapy	0 20
Hemp on oak, dry	0 53
Hemp on oak, wet	0 33
Metals on metals, dry	0 15–0 20
Steel on agate, dry	0 20
Steel on agate, oiled	0 11
Smooth, hard oiled surfaces	0 05

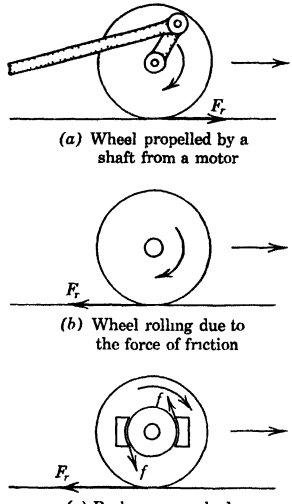
In recent years many experiments have been made in order to further our understanding of the mechanism of the force of friction. By means of interference, the microscope, and the electron microscope it has been shown that the best flat surface has hills and valleys on it that are large compared to molecular dimensions. Even with the most careful preparation it is difficult to obtain a surface that is flat to within 100 to 1000 Angstroms (1 Angstrom = 10^{-8} cm). The question then arises as to what is the "real" area of contact between two surfaces? This has been estimated by measuring the electrical resistance between the surfaces of contact of metals. It has been found that the real area is almost independent of the actual area of the surfaces and is approximately proportional to the load pressing the two surfaces together. These results do not affect the laws of friction given above since the actual area of contact is used in stating them.

The real area of contact is a very small fraction of the actual area, and the force per unit area or pressure at these points of contact may

be sufficiently high to cause localized flow in metals. There are localized adhesions such that there is transfer of metal at the sliding surfaces. If the slider surface is made radioactive, it has been found that small portions of the radioactive material are transferred to the other surface. It might be thought that these localized adhesions would cause the surfaces to adhere when the load was removed. This is found to be so for metals from which all surface oxide layers have been removed by heating in a vacuum. For hard metals in air it appears that although the metal is deformed plastically at the points of contact there is elastic deformation in the bulk of the metal nearby. When the load is removed, the elastic stresses break any junctions that have been formed.

By impregnating the surface layers of sintered copper with the plastic Teflon, a material has been made with a coefficient of friction of about 0.05. This is independent of temperature up to 250°C, and it appears that copper-impregnated Teflon would be a valuable antifriction material.

It is interesting to consider how the force of friction between the wheels of a locomotive and the ground changes during the operations of propulsion and braking.



(*a*) Wheel propelled by a shaft from a motor

(*b*) Wheel rolling due to the force of friction

(*c*) Brakes are applied to a wheel

FIG. 2 10 The frictional force F_r on a wheel under different conditions.

Case A. Consider a locomotive or automobile wheel driven by some mechanism from a motor. If the wheel is to move forward on the ground the instantaneous point of contact between the wheel and the ground must be at rest. For this to be so a frictional force in the direction of motion must act on the wheel, Fig. 2.10*a*.

Case B. Consider the wheels of a locomotive or automobile running freely without any driving mechanism. Without friction the wheels would slide along the ground. It is the force of friction which causes the wheels to rotate. The force of friction on the wheels is then in the opposite direction to the motion, Fig. 2.10*b*.

Case C. Consider the effect of brakes in stopping a car. Brakes exert forces f on the moving wheel, tending to stop its rotation, Fig.

2.10c. If the forward motion of the wheel is to be stopped, that is, if there is to be a backward acceleration, then the frictional force must act in the backward direction. It is the braking force which stops the wheels from rotating; it is the force of friction between the wheels and the ground which stops the car. If the forces f exerted by the brakes are very large compared to the frictional force between the wheels and the road, it is possible to have the wheels lock and then skidding takes place.*

2.11 Motion of Projectiles

Here the problem is to find the time of flight and range on a horizontal plane of a particle initially projected at some angle with the horizontal. Suppose that a particle of mass m is projected with a

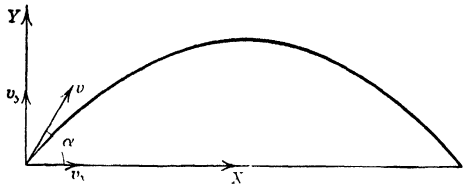


FIG. 2.11 Parabolic path of a projectile, assuming no air resistance.

velocity v at an angle α with the horizontal, Fig. 2 11. We shall make the following assumptions in this problem:

1. The value of the gravitational acceleration g is constant throughout the motion.
2. The forces due to air resistance can be neglected.
3. Any effects due to the rotation of the earth may be neglected.

If F_x is the component of force acting on the particle along the X axis and F_y is the component along the Y axis, then, taking into consideration these three assumptions and Newton's second law, the equations of motion of the particle are

$$F_x = mx = 0$$

$$F_y = my = -mg$$

Integrating the first of the equations gives

$$\dot{x} = C_1 \qquad \text{and} \qquad x = C_1 t + C_2$$

The initial velocity of the particle along the X axis is the resolved part

* For a detailed discussion of the laws of friction see the articles by F. Palmer, *Am. J. Physics*, **17**, 327, 1949.

of the velocity v along this direction. Thus at $t = 0$

$$\dot{x} = v \cos \alpha$$

Hence the constant C_1 is $v \cos \alpha$; i.e., the horizontal velocity is constant. If the origin of the coordinates is at the point of projection, then, at $t = 0$, $x = 0$ and the constant $C_2 = 0$. The horizontal distance traveled in time t is

$$x = vt \cos \alpha \tag{2.7}$$

For motion in the positive Y direction we have

$$y = -g \qquad \dot{y} = -gt + C_1$$

From the initial conditions of motion, $\dot{y} = v \sin \alpha$ when $t = 0$. Hence

$$C_1 = v \sin \alpha$$

or

$$\dot{y} = -gt + v \sin \alpha \tag{2.8}$$

Integrating again gives

$$y = -\frac{gt^2}{2} + vt \sin \alpha + C_2 \tag{2.9}$$

At $t = 0$, $y = 0$ Hence $C_2 = 0$.

The equation of the path traveled by the particle is obtained by eliminating the time t between Eqns. 2.7 and 2.9, giving

$$y = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha} \tag{2.10}$$

Since for any particular projection v, g, and α are constants, it follows that Eqn. 2.10 represents a parabola

The time t_f taken for the projectile to attain its maximum height is the time taken for the vertical velocity to become zero. From Eqn. 2.8

$$t_f = \frac{v \sin \alpha}{g}$$

The total time the particle is in flight is $2t_f$, since the time of descent is equal to that of ascent During this time the particle has been moving with a constant horizontal component of velocity $v \cos \alpha$ so that the total horizontal distance traveled by the particle, or its horizontal range, is

$$2tv \cos \alpha = \frac{2v^2 \sin \alpha \cos \alpha}{g} = \frac{v^2 \sin 2\alpha}{g}$$

The maximum range occurs when sin 2α has its maximum value of unity, that is, when

$$2\alpha = 90^\circ \qquad \text{or} \qquad \alpha = 45^\circ$$

Thus for any angle of projection greater or less than 45° the range is smaller than the maximum value.

To determine the range of a projectile on an inclined plane it is simplest to find the point of intersection of the inclined plane and the parabolic path of flight. Let the point of intersection be P having

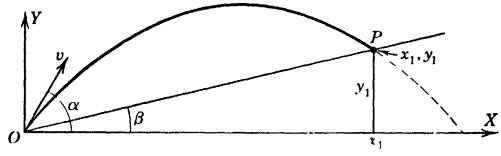


FIG. 2.12 The range of a projectile on an inclined plane.

coordinates x_1, y_1 as in Fig. 2.12. The angle of projection with the horizontal is α, and the angle of inclination of the plane is β. Since x_1, y_1 is a point on the flight path, it follows from Eqn. 2.10 that

$$y_1 = x_1 \tan \alpha - \frac{g{x_1}^2}{2v^2 \cos^2 \alpha}$$

From Fig. 2.12 it may be seen that

$$y_1 = x_1 \tan \beta$$

Thus the coordinate x_1 is given by

$$x_1 \tan \beta = x_1 \tan \alpha - \frac{g{x_1}^2}{2v^2 \cos^2 \alpha}$$

or

$$x_1 = \frac{2v^2}{g} \cos^2 \alpha (\tan \alpha - \tan \beta)$$

The range OP on the inclined plane is

$$OP = \frac{x_1}{\cos \beta} = \frac{2v^2 \cos^2 \alpha}{g \cos \beta} (\tan \alpha - \tan \beta) \qquad (2.11)$$

2.12 The Inclined Plane with Friction

Suppose that a body of mass m_1 is placed on an inclined plane whose angle of inclination is α. From the mass m_1 a cord is attached which

passes over a smooth peg to a hanging mass m_2, Fig. 2.13. The coefficient of friction between the mass m_1 and the plane is μ. In any numerical problem one first has to determine the direction of motion of the masses. Depending on the relative values of m_1 and m_2 and the value of μ, there may be either no motion or an acceleration up or down the plane. To be specific let us assume that the mass m_1 is

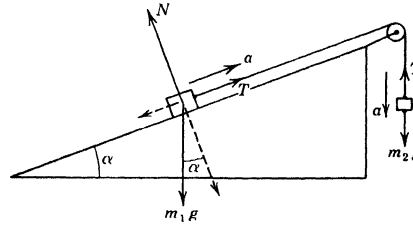


FIG. 2.13 Motion on an inclined plane.

accelerated up the plane with an acceleration a. Then, assuming that the cord is inextensible, the mass m_2 has the same acceleration vertically downward. The force of friction which opposes the motion is

$$\mu N = \mu m_1 g \cos \alpha$$

By Newton's second law the equation of motion of m_1 is

$$T - m_1 g \sin \alpha - \mu m_1 g \cos \alpha = m_1 a$$

and of m_2 is

$$m_2 g - T = m_2 a$$

Hence

$$a = \frac{(m_2 - m_1 \sin \alpha - m_1 \mu \cos \alpha) g}{(m_1 + m_2)}$$

2.13 Motion in a Resisting Medium

When an object moves through a medium such as air, it is subjected to a resistance force. This resistance force is a function of the velocity and the size and shape of the moving object. There is no simple relationship between the velocity and the force, but at very low velocities it appears that the force varies approximately as the velocity whereas at higher velocities it appears to be approximately proportional to the square or even higher powers of the velocity. In any practical situation the relationship between the resistance force and the velocity has to be obtained experimentally.

Without air resistance a parachute would be useless. When a man jumps from a plane with his open parachute, he is accelerated vertically until he reaches such a velocity that the air resistance force exactly equals the total weight of man and parachute. This maximum velocity is called the *terminal velocity* of the system. Suppose that the mass of the falling object is m and at any particular velocity v the air resistance force is mkv, where k is a constant of proportionality with dimensions $[T^{-1}]$ At the terminal velocity v_t the resistance force is exactly equal to the weight:

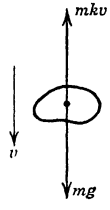


FIG. 2.14 Motion against air resistance.

$$mkv_t = mg$$

or

$$v_t = \frac{g}{k} \tag{2.12}$$

For a downward velocity v less than v_t the body has a downward acceleration dv/dt and the equation of motion is, from Fig. 2.14,

$$mg - mkv = m\frac{dv}{dt}$$

where the downward direction is taken as positive. Thus

$$\frac{dv}{g - kv} = dt$$

If we assume the acceleration of gravity g to be constant, the integration of this equation gives

$$-\frac{1}{k}\log_e(g - kv) = t + C$$

Suppose that the body starts from rest. Then, at $t = 0$, $v = 0$ and $C = -1/k \log_e g$, so that

$$-\frac{1}{k}\log_e\left(\frac{g - kv}{g}\right) = t$$

or

$$g - kv = ge^{-kt}$$

The velocity v at any time t is

$$v = \frac{g}{k}(1 - e^{-kt}) \tag{2.13}$$

Thus the velocity starts from zero and increases until after an infinite

time, when the exponential term becomes zero, the velocity reaches a constant or terminal value of v_t, as shown in Fig. 2.15.

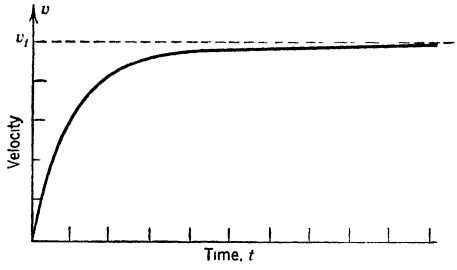


FIG. 2.15 Velocity plotted against time for motion in a resisting medium.

Suppose that the body starts from some height h above the ground. The velocity v may be written as dy/dt where y is measured downward with its origin at the point of fall. Thus from Eqn. 2.13

$$\frac{k}{g}dy = (1 - e^{-kt})\,dt$$

and by integration

$$\frac{k}{g}y = t + \frac{e^{-kt}}{k} + C$$

Since $y = 0$ at $t = 0$,

$$C = -\frac{1}{k}$$

and the distance y fallen through in time t is given by

$$\frac{k}{g}y = t + \frac{e^{-kt}}{k} - \frac{1}{k}$$

or

$$y = \frac{g}{k^2}(e^{-kt} + kt - 1) \tag{2.14}$$

If the body started from a height h above the ground and reached the ground in time T, then

$$h = \frac{g}{k^2}(e^{-kT} + kT - 1)$$

or the time of fall T is

$$T + \frac{e^{-kT}}{k} = \frac{kh}{g} + \frac{1}{k} \tag{2.15}$$

Possibly the simplest way to solve for T is to proceed by trial and error. Suppose that we wish to calculate the time of fall of a body from a height of 10,000 ft with the value of the air resistance constant equal to 0.1. If there were no air resistance the time of fall would be given by the equation

$$h = \tfrac{1}{2} gt^2$$

or

$$t^2 = \frac{2 \times 10^4}{32} \qquad \text{giving } t = 25 \text{ sec}$$

With air resistance the time of fall would be larger. As a first guess let us take the time T as 40 sec. Then $40 + 10e^{-4}$ should be equal to $(10^3/32) + 10$, or 41.25. From a table of exponentials the left-hand side of Eqn. 2.15 is found to equal 40.18 which is less than the number of seconds required. Since the exponential term yields only a small number, then for a second guess let us take $T = 41.1$ sec. With this value for T the left-hand side is

$$41.1 + 10e^{-4\ 11} = 41.1 + 0.16 = 41.26$$

This is sufficiently close to the value of 41.25 on the right-hand side to be acceptable.

If the particle were projected vertically upward with an initial velocity V and were subjected to a resistance force of mkv, the equation of motion would be

$$m\frac{dv}{dt} = -mg - mkv \qquad \text{or} \qquad \dot{y} = -g - k\dot{y}$$

where v is the velocity at any time t. In this example the origin is taken at the ground and the upward direction is taken as positive This equation can be integrated if the initial conditions are known. This would give the maximum height attained by the particle and also the time taken to reach the maximum height. The result of a similar integration is given by Eqn. 2.17.

Example of a projectile moving in a resisting medium. If a projectile is shot with a velocity V at an angle α with the horizontal and is subject to a retardation of kv due to air resistance, then it no longer follows a parabolic path. The equations of motion along the horizontal or X axis and the vertical or Y axis are

$$m\ddot{x} = -mk\dot{x} \qquad \text{or} \qquad \ddot{x} = -k\dot{x}$$

and

$$m\ddot{y} = -mg - mk\dot{y} \qquad \text{or} \qquad \ddot{y} = -g - k\dot{y}$$

Integrating the first of these equations with the conditions that, at time $t = 0$, $\dot{x} = V \cos \alpha$, we have

$$\int \frac{d\dot{x}}{\dot{x}} = \int -k\, dt + C$$

or

$$\log_e \dot{x} = -kt + \log_e (V \cos \alpha)$$

or

$$x = (V \cos \alpha)e^{-kt}$$

The distance traveled in a time t is by integration

$$x = \frac{-V \cos \alpha}{k} e^{-kt} + C$$

If $x = 0$ at $t = 0$, then

$$C = \frac{V \cos \alpha}{k}$$

and

$$x = \frac{V \cos \alpha}{k} (1 - e^{-kt}) \tag{2.16}$$

The elevation y of the projectile at any time t can be found by integration to be

$$y = \frac{1}{k^2} (g + kV \sin \alpha)(1 - e^{-kt}) - \frac{gt}{k} \tag{2.17}$$

since the initial velocity is $V \sin \alpha$. Eliminating the time t between Eqns. 2.16 and 2.17, we obtain the equation of the path as

$$y = \frac{1}{k^2} (g + kV \sin \alpha) \left(\frac{kx}{V \cos \alpha} \right) - \frac{g}{k^2} \log_e \left(\frac{V \cos \alpha}{V \cos \alpha - kx} \right)$$

or

$$y = \frac{x}{V \cos \alpha} \left(V \sin \alpha + \frac{g}{k} \right) + \frac{g}{k^2} \log_e \left(1 - \frac{kx}{V \cos \alpha} \right) \tag{2.18}$$

If the right-hand side of this equation is expanded in powers of k, then

$$y = \frac{x}{V \cos \alpha} \left(V \sin \alpha + \frac{g}{k} \right) + \frac{g}{k^2} \left(- \frac{kx}{V \cos \alpha} - \frac{k^2 x^2}{2V^2 \cos^2 \alpha} - \frac{k^3 x^3}{3V^3 \cos^3 \alpha} - \cdots \right)$$

or

$$y = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha} - \frac{gkx^3}{3V^3 \cos^3 \alpha} - \frac{gk^2 x^4}{4V^4 \cos^4 \alpha} - \cdots \tag{2.19}$$

which, when $k = 0$, reduces to Eqn. 2 10, the equation for the parabolic path of a projectile without air resistance. The path of the projectile given by Eqn. 2.19 is no longer parabolic since for each value of k there is a differently shaped path. It can be shown that the angle at which the projectile strikes the ground is greater than the angle of projection α. The slope of the path at any point is given by the value of dy/dx at that point. Thus it is readily seen that at the point of projection, $x = 0$,

$$\left(\frac{dy}{dx} \right)_{x=0} = \tan \alpha$$

Although the assumption that the resistance force is proportional to the velocity is not completely justified, in practice this analysis does give a first approximation to the path followed by a projectile in air.

In this chapter we have discussed the action of forces on a particle and the resulting acceleration. To do this we have isolated the particle and listed all the forces acting on it before applying Newton's laws. It cannot be overemphasized that a carefully drawn diagram listing the forces and the directions of the velocities or accelerations is of great value.

PROBLEMS

1. If a force of a 100 newtons acts on a mass of 100 gm, find the acceleration in meters per second per second.

2. A force of 10 lbw acts on a body weighing 16 lb. Find:

a. The acceleration, using the British engineering system of units.

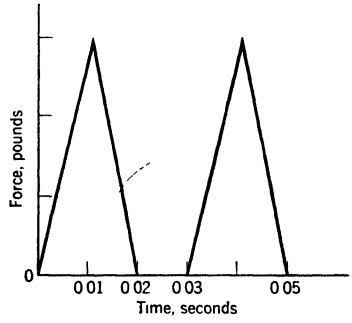
b. The number of poundals in 10 lbw.

c. The acceleration, using the absolute system of units.

3. A cord having zero mass passes over a frictionless and massless pulley. On one side hangs a monkey, and on the other a counterbalance. If the monkey starts to climb up the cord with an acceleration, show that the counterbalance moves upward with the same acceleration.

4. An Atwood machine has weights of 10 lbw and 8 lbw respectively hanging on the ends of a cord passing over a pulley. Neglecting the masses of the cord and the pulley and assuming that there is no friction, find the acceleration of the system and the tension in the cord.

5. Machine-gun bullets are fired at a target in a regular manner, as shown in the accompanying figure. Find the average value of the force exerted by the bullets on the target, assuming the maximum force is 300 lbw



PROBLEM 5

6. A body having a weight of 1 kg is placed on an inclined plane having an angle of inclination of 30°. Find:

a. The acceleration of the weight down the plane if the plane is smooth.

If the coefficient of friction between the weight and the plane is 0.2, find:

b. The acceleration of the weight down the plane.

c. The smallest value of the coefficient of friction that will just prevent any sliding.

7. Two cars A and B are traveling toward an intersection and have a collision. Car A has a weight of 3000 lb and is moving with a speed of 40 mph toward the west; car B has a weight of 4000 lb and a speed of 50 mph toward the south. If they interlock at the collision, find the magnitude and direction of the velocity of the cars immediately after collision.

8. A bullet is shot at an angle of 60° with the horizontal with an initial speed of 2000 ft per sec. Neglecting air resistance, find:

a. The maximum elevation of the bullet

b. The range of the bullet along a horizontal plane.

9. A bullet is shot at an angle of 60° with the horizontal along the line of maximum slope of a hill whose angle of inclination is 10°. If the bullet has an initial velocity of 2000 ft per sec, find the range of the bullet along the hill.

10. A bullet is shot at an angle α with the horizontal along the line of maximum slope of a hill whose angle of inclination is β. If the bullet has an initial velocity of v, show that when air resistance is neglected the range of the bullet on the hill is

$$R = \frac{v^2}{g} \sec^2 \beta[\sin (2\alpha - \beta) - \sin \beta]$$

and that the range has a maximum value when

$$\alpha = 45° + \frac{\beta}{2}$$

and that this maximum range is

$$\frac{v^2}{g(1 + \sin \beta)}$$

If the bullet strikes the inclined plane at an angle ϕ with the horizontal, show that

$$\tan \phi = \left(\frac{dy}{dx}\right) = 2 \tan \beta - \tan \alpha$$

11. A bullet is shot with an initial speed of 1000 ft per sec at an angle of 60° with the horizontal from the top of a hill whose angle of inclination with the horizontal is 10°. If the bullet is shot along the line of greatest slope, show that its range down the hill is approximately 30,250 ft.

12. A man having a weight of 200 lb stands on an accurate weighing machine in an elevator. Find the reading of the weighing machine when the elevator is:

a. Ascending with a constant speed of 10 ft per sec.

b. Ascending with an upward acceleration of 4 ft per sec per sec.

c. Descending with a downward acceleration of 4 ft per sec per sec.

d. If the elevator has a speed of 10 ft per sec at the fifteenth floor and comes to rest at the twentieth floor in a distance of 50 ft, find the reading of the weighing machine during this deceleration.

e. Find the acceleration of the elevator when the reading of the weighing machine is 190 lb.

13. A bullet having a mass of 4 gm and a velocity of 2000 cm per sec penetrates 2 cm into a block of wood before being brought to rest.

a. Find the average resistance force exerted by the wood on the bullet

b. If the piece of wood were only 1.5 cm thick and the resistance force the same as in part *a*, find the velocity with which the bullet emerges.

14. A 5-lb block is started from the bottom of an inclined plane with a speed of 50 ft per sec. If the inclined plane has an angle of inclination of 30° and the coefficient of friction between the block and the plane is 0.2, find:

a. How far up the plane the block moves before coming to rest.

b. The speed of the block when it reaches the bottom of the plane again.

15. A block slides without friction from the top of a roof 18 ft long which is inclined at an angle of 30° with the horizontal. If the edge of the roof is 10 ft above the ground, find how far the block strikes the ground from the point directly below the edge of the roof.

16. A ball A is projected at an angle of 60° with the horizontal towards a ball B, and at the instant the ball A is projected the ball B starts to fall. Prove that the two balls will collide if the velocity of projection is large enough. If the initial distance between the balls is 8 ft and the initial velocity of A is 16 ft per sec at an angle of 60° with the horizontal, find:

a. The length of time after projection when the balls collide

b. The magnitude and direction of the velocity of ball A just before it collides with ball B.

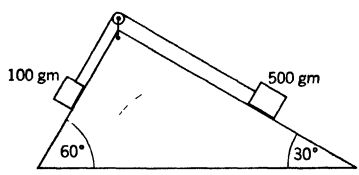
17. A 60-gm block is placed on an inclined plane whose angle of inclination is 30°. Attached to the 60-gm block is a cord which passes over a smooth peg at the top of the plane and which has a mass of 50 gm hanging vertically from it Find:

a. The maximum value of the coefficient of friction such that the 60-gm block will just move up the plane

b. The acceleration of the system and the tension in the cord when the coefficient of friction is 0 10

c. The motion when the coefficient of friction is 0.40.

18. Masses of 100 and 500 gm are connected by a cord which passes over a smooth peg as shown in the figure. Find the acceleration of the system and the tension in the cord if the coefficient of friction is 0.20.



PROBLEM 18

19. A bullet whose mass is 3 gm has a velocity of 20,000 cm per sec and is horizontally fired into a block of wood of mass 500 gm lying at rest on a horizontal table. If the coefficient of friction between the block and the table is 0.30 and the bullet becomes embedded in the block, find:

a. The velocity of the bullet and the block immediately after the bullet comes to rest in the block

b. The distance moved by the block before it comes to rest.

If the bullet penetrates 5 cm into the block of wood, find:

c. The force exerted by the bullet on the block.

20. A bullet is shot vertically upward with an initial velocity of 1000 ft per sec.

If it is subjected to an air resistance force of mkv where k has the value of 0.10, m is the mass, and v the velocity of the bullet, find:

a. How high it rises

b. How long it takes to reach its maximum altitude.

c. The velocity with which it strikes the ground.

d. How long it takes to get from its maximum altitude to the ground.

21. Repeat problem 20 for a resistance force of mkv^2 where k again has the value of 0.10.

22. A bullet is shot with an initial velocity of 1000 ft per sec at an angle of 30° with the horizontal. If the air resistance force is mkv where k has the value of 0 10, find:

a. The maximum altitude attained

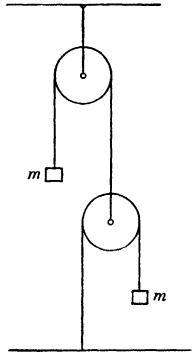
b. The total time of flight

c. The range of the bullet on a horizontal plane.

23. A particle is shot vertically upwards with an initial velocity of u. If the retardation due to the air is equal to a constant k times the square of the velocity of the particle, show that the particle will reach the ground with a velocity v given by:

$$\frac{1}{v^2} = \frac{1}{u^2} + \frac{k}{g}$$

24. A pulley system is set up as shown in the figure with equal masses on the ends of the cord. If each pulley has a negligible mass, show that the acceleration of the mass on the left is $g/5$ while the acceleration of the mass on the right is $2g/5$.



PROBLEM 24

25. If the movable pulley in problem 24 has the same mass as the hanging ones, show that the acceleration of the mass on the left is $g/3$ and on the right is $2g/3$.

3 · WORK AND ENERGY OF PARTICLES

3.1 Introduction

The concept of energy is one of the most important in physics, with applications in every branch of science. We are so familiar with its manifestations and transformations in our daily life that they often evade our notice. Energy is now recognized in a great variety of forms—gravitational, electrical, chemical, thermal, atomic, etc. The standard of living of man has always been dependent upon the extent to which he could utilize the various forms of energy. With the discovery by Einstein in 1905 that matter itself is a form of energy, science has progressed until our very civilization may be threatened by its uncontrolled use for military purposes.

It is about a hundred years since the principle of conservation of energy was first announced. This principle states that energy can never be destroyed; it can only be changed from one form to another. Though many new forms of energy have been recognized in the past one hundred years, the principle of conservation of energy has never been invalidated.

The energy of a body is measured by the work the body is capable of doing. Work is done whenever a force moves a body; it is measured by the product of the force and the distance moved in the direction of the force. Though force and displacement are each vector quantities, work and energy are scalars with magnitude but not direction. Since we are not concerned with energy of rotation we shall consider bodies to be particles in this chapter.

3.2 Definitions, Units, and Dimensions of Work

Suppose that a force F acts on a particle and moves it a distance dr in such a direction that the angle between the force and the displacement is θ, Fig. 3.1. Then the work W done by the force on the particle is

$$W = F \cos \theta \, dr \tag{3.1}$$

Since the quantity $F \cos \theta$ is the component of the force in the direction of the displacement dr, *the work done by any force is the product of the*

magnitude of the displacement and the magnitude of the component of the force in the direction of the displacement.

If the force is at right angles to the displacement, no work is done by the force in this displacement. Thus no work is done by or against the force due to gravity when a body is moved in a horizontal direction. The work done by a force on a body can be positive, zero, or negative, according to whether the cosine term is positive, zero, or negative. Work is positive when the force has a component in the direction of the displacement, zero when the force is at right angles to the displacement, and negative when the force has a component in the direction opposite

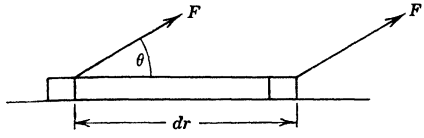


FIG. 3.1 Force F moving an object through a distance dr.

to the displacement. As an example of negative work consider a body being moved along a rough horizontal plane. Work is done by the forward force against the backward force of friction. Thus the work done by the force of friction during the displacement is a negative quantity.

The common unit of work in the cgs system is the dyne centimeter and is called the *erg*. One erg of work is done on a body when a force of 1 dyne moves the body through a distance of 1 cm in the direction of the force. In the mks system the unit of work is the newton meter which is called a *joule*. Thus 1 joule of work is done on a particle when a force of 1 newton moves the particle through a distance of 1 meter in the direction of the force. Since 1 newton is equal to 10^5 dynes and 1 meter is equal to 10^2 cm, it follows that

$$1 \text{ joule} = 10^7 \text{ ergs}$$

In the metric gravitational system a unit of work is 1 gmw multiplied by 1 cm, or 1 gmw cm, where approximately

$$1 \text{ gmw cm} = 980 \text{ ergs}$$

In the British gravitational or engineering system the unit of work is 1 lbw of force multiplied by 1 ft, 1 ft lbw. Thus 1 ft lbw of work is done on a particle when a force of 1 lbw moves the particle through a distance of 1 ft in the direction of the force.

The dimensions of work in the metric absolute system are $[MLT^{-2}L]$

or $[ML^2T^{-2}]$. In the engineering or gravitational system force, length, and time are arbitrarily taken as fundamental, so the dimensions of work in this system are $[FL]$.

3.3 Linear Kinetic Energy

Let us now consider the work done by a horizontal force moving a particle a horizontal distance. Since we are discussing linear and not rotational motion, this may be considered to be an automobile accelerating along a horizontal road. The forward force F is the resultant of a forward force F_1 and a backward resistance force F_2. Both these forces may be considered to be horizontal so that

$$F = F_1 - F_2$$

Work is done by the forward force F_1 and against the backward force F_2 during the displacement. The resultant force F gives the automo-

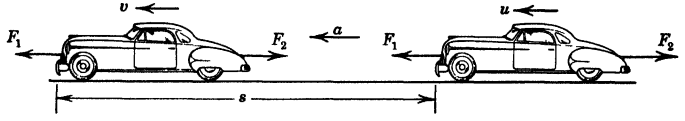


FIG. 3.2 Kinetic energy given to an automobile.

bile an acceleration, and its speed is increased. Thus the work done by the resultant force goes into increasing the speed of the automobile or giving the automobile kinetic energy.

Suppose that the automobile has a mass m and an initial velocity u when the force F starts to act, so that the automobile has a constant acceleration a, and after traveling a distance s its velocity is v, Fig. 3.2. From Eqn. 1.2 it follows that

$$v^2 - u^2 = 2as$$

The acceleration a of the automobile is given by Newton's second law as

$$F = ma$$

and the work done by the horizontal force F in acting through a horizontal distance s is

$$\begin{aligned} Fs = mas &= \frac{ma}{2a}(v^2 - u^2) \\ &= \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 \end{aligned} \tag{3.2}$$

The quantity $\frac{1}{2}mv^2$ is by definition the *linear or translational kinetic energy, K.E.*, of a particle of mass m moving with a linear velocity v.

Thus the work done by the accelerating force F on the car is equal to the increase in kinetic energy of the car. To stop the car a braking or decelerating force is necessary. The work done by the braking force on the car is equal to its decrease in kinetic energy. In order to increase or decrease the kinetic energy of a particle an accelerating or decelerating force must act through some distance.

The units and dimensions of kinetic energy are the same as those of work as may be easily shown. In the British engineering system of units the kinetic energy equation is written

$$Fs = \frac{1}{2}\frac{w}{g}(v^2 - u^2)$$

where w is the weight of the accelerated particle and F the resultant force acting on the particle. Both F and w are in the same units, namely pounds weight.

A more general and sophisticated manner of deriving the relationship between work done and kinetic energy will now be given. If a particle of mass m, acted on by a resultant force $\mathbf{F}$, moves a distance $d\mathbf{r}$ in a direction making an angle θ with the force, then the work done by the force is by Eqn. 3.1

$$W = F \cos \theta \, dr$$

This can be written in vector form as is done in section 1.8. Thus

$$W = \mathbf{F} \cdot d\mathbf{r}$$

This is called the dot or scalar product of the vectors $\mathbf{F}$ and $d\mathbf{r}$. The expression $\mathbf{F} \cdot d\mathbf{r}$ is read as F dot dr and by definition has a magnitude equal to the product of the magnitudes of the two vectors and the cosine of the angle between their positive directions.

An analytical expression for the dot product can be written by expressing the force $\mathbf{F}$ and the displacement $d\mathbf{r}$ in terms of their respective components and the unit vectors along the X, Y, Z axes. If F_x, F_y, F_z are the respective magnitudes of the force $\mathbf{F}$ along the X, Y, Z axes, then from Eqn 1.6 it follows that

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

Similarly if dx, dy, dz are the magnitudes of the components of the displacement dr, then

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$$

Thus

$$\mathbf{F} \cdot d\mathbf{r} = (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

This expression can be simplified by recognizing that

$$\mathbf{i} \cdot \mathbf{i} = \cos 0° = 1 = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k}$$

and

$$\mathbf{i} \cdot \mathbf{j} = \cos 90° = 0 = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i}$$

Hence by multiplication

$$\mathbf{F} \cdot d\mathbf{r} = F_x\, dx + F_y\, dy + F_z\, dz \tag{3.3}$$

The quantity $F_x\, dx$ is the work done by the component F_x moving a distance dx. Since both these quantities are scalars, it follows that the work $F_x\, dx$ is also a scalar quantity. Thus the work $\mathbf{F} \cdot d\mathbf{r}$ is a scalar quantity. Later we shall see that two vectors can be multiplied to give a vector or cross product.

Let us return now to the particle of mass m acted on by a resultant force **F**. Its acceleration is given by

$$\mathbf{F} = \frac{m\, d^2\mathbf{r}}{dt^2} = m\ddot{\mathbf{r}}$$

The work done by the force **F** in the displacement $d\mathbf{r}$ is

$$W = \mathbf{F} \cdot d\mathbf{r} = m\ddot{\mathbf{r}} \cdot d\mathbf{r}$$

From the identity

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}\, dt = \dot{\mathbf{r}}\, dt \tag{3.4}$$

the acceleration equation may be changed into a velocity equation Thus

$$\mathbf{F} \cdot d\mathbf{r} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}\, dt$$

$$= \frac{m}{2} \frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt}\, dt$$

$$= \frac{m}{2}\, d(\dot{r}^2)$$

If the velocity of the particle is u at time t_0 and position $\mathbf{r}_0$ and is v at a later time t and position $\mathbf{r}$, then by integration

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \int_u^v \frac{m}{2}\, d(\dot{r}^2) = \frac{m}{2} v^2 - \frac{m}{2} u^2 \tag{3.5}$$

Thus the work done by the resultant force **F** during the time $(t - t_0)$ in which there is a displacement of $(\mathbf{r} - \mathbf{r}_0)$ is equal to the change in kinetic energy of the particle.

3.4 Potential Energy

A body at rest may possess energy owing to its position or configuration. A raised weight and an extended or compressed spring are examples of bodies possessing energy owing to their position; that is, they possess potential energy. Consider a body of mass m raised a vertical height y from position A to B, Fig. 3.3. The work done against the force of gravity in lifting the body from A to B is mgy since the force due to gravity mg acts vertically downward and the upward vertical direction is taken as positive. In position B the mass m possesses gravitational potential energy mgy relative to position A. In other words, the difference in potential energy of the mass m in positions B and A is mgy. It is assumed that the body is at rest at A and B and that any displacement is made very slowly so that the kinetic energy is negligible. Gravitational potential energy increases upward while the force of gravity acts downward.

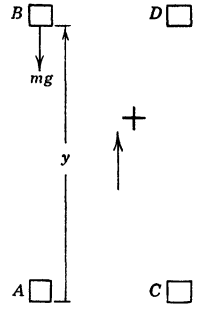


FIG. 3.3 Gravitational potential energy.

Let us now consider the work done against gravity when the mass m is taken along the rectangular path $ACDB$ from A to B. Along the horizontal paths AC and DB no work is done by or against the force of gravity since the displacements are horizontal. The only work done against gravity along the path $ACDB$ is mgy. No matter what path is chosen to take the mass m from A to B, it may be broken up into a series of horizontal and vertical paths. The total vertical displacement will be AB or y, and the total work done against gravity mgy. When the mass m is moved from B back to A, an amount of work mgy is done by the force of gravity. There is a decrease in the potential energy of the mass m equal to mgy. If the work done against gravity is taken as positive, then the work done by gravity must be negative. Therefore the total work done against or by gravity when a body is taken around a closed path is zero. Any force of such a nature that the total work done against it in a closed path is zero is called a *conservative force*. The force of gravity is such a force. There are other conservative forces, a familiar one being the force exerted by an electrostatic field on an electrically charged body.

The difference in potential energy of a body moved between two points in a conservative force field is the work done against the conservative force, or the negative of the work done by the conservative force, when the body is moved between the two points. Thus, if F_y

is a conservative force, then, when a body is moved against this conservative force from position y_0 to y, the increase in potential energy V is given by

$$V = -\int_{y_0}^{y} F_y \, dy \tag{3.6}$$

and the conservative force F_y is

$$F_y = -\frac{dV}{dy} \tag{3.7}$$

In the gravitational case the difference in potential energy of a mass m moved a vertical distance y is

$$V = mgy$$

and

$$F_y = -\frac{dV}{dy} = -mg$$

The force F_y is positive in an upward direction, that is, in the positive direction of y, while $-F_y$ which is equal to mg acts downward. This of course is in accord with Eqn. 3.6.

If the mass m in Fig. 3.3 is allowed to fall freely from B to A, then the work done by the conservative force, that is, the decrease in potential energy, is converted into kinetic energy. The decrease in potential energy is

$$V_B - V_A = mg(BA)$$

From Eqn. 3.2 this is equal to the change in kinetic energy

$$V_B - V_A = \tfrac{1}{2}mv_A^2 - \tfrac{1}{2}mv_B^2$$

Hence

$$V_B + \tfrac{1}{2}mv_B^2 = V_A + \tfrac{1}{2}mv_A^2$$

Thus the sum of the potential and kinetic energies of the mass at B is equal to the sum of these energies at A. This situation applies to any point along the path followed by the body. In this motion there is *conservation of the mechanical energy of the body.* Mechanical energy exists in the two forms, kinetic and potential. If the mass starts from rest at B, then

$$V_B - V_A = mg(BA) = \tfrac{1}{2}mv_A^2$$

or

$$v_A^2 = 2g(BA)$$

This same result could have been obtained from the kinematical equa-

tions in Chapter 1 for a body falling from rest through a height BA with the constant acceleration g due to gravity.

The law of conservation of mechanical energy may be derived in a more general manner. Consider that a conservative force **F** moves a body from position A to position B, Fig. 3.4. If V_A, V_B are the

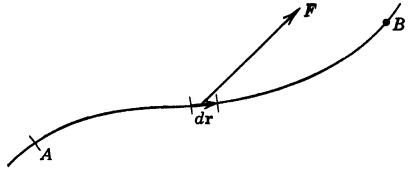


FIG. 3.4 Work done by a conservative force in moving a body between two points is independent of the path.

respective potential energies of the body at A and B, then by Eqn. 3.6 it follows that the work done by the force **F** is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B -dV = V_A - V_B \tag{3.8}$$

Also, if the conservative force **F** is the resultant force acting on the mass m and the velocity at A is v_A and at B is v_B, then by Eqn. 3.5

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

Thus

$$V_A - V_B = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

or

$$V_A + \tfrac{1}{2}mv_A^2 = V_B + \tfrac{1}{2}mv_B^2$$

If, as is frequently done, the symbol T indicates kinetic energy, then the law of conservation of mechanical energy may be written

$$V_A + T_A = V_B + T_B$$

or

$$V + T = \text{a constant} \tag{3.9}$$

At any point along the path of a particle moving in a conservative force field the sum of the kinetic and potential energies of the body is a constant.

If other forces such as air resistance or friction are present, the total work done on a body in taking it around a closed path is no longer zero. Resistance forces always oppose motion, and work done against them goes into heat energy. Such forces are called dissipative forces

and are not capable of being derived from a potential function. When dissipative forces are present, there is no longer a conservation of mechanical energy. A decrease in potential energy does not yield a corresponding increase in kinetic energy for some of the energy goes into heat.

We shall now generalize the expression given in Eqn. 3.7 and show that the space rate of change of potential energy in any direction is equal to the negative of the conservative force in that direction. Suppose that a force **F** displaces a particle through a distance $d\mathbf{r}$. Then the work done is

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

This may be written in terms of the components in the X, Y, Z directions as in Eqn. 3.3 so that

$$\mathbf{F} \cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz$$

Suppose now that the components F_x, F_y, F_z of the force **F** are functions only of the coordinates x, y, z of the point at which the force is being considered, so that

$$F_x\,dx + F_y\,dy + F_z\,dz = -dV(xyz)$$

where dV is a perfect differential. If dV is a perfect differential, it follows that

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz$$

Thus

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z} \tag{3.10}$$

Since the function V is the potential energy of the particle, it follows that *the component of the conservative force in any direction is equal to the space rate of decrease of potential in that direction.* Partial derivatives are employed since the function V depends on the three coordinates x, y, z, and in taking the derivative with respect to one coordinate the other two coordinates are considered constant.

This result may become somewhat clearer if we consider a contour map, Fig. 3.5, on which the closed curves represent places of equal elevation above sea level. They are curves of equal gravitational potential energy or equipotential curves. If you have carried such a map when walking through mountainous country you have probably realized, sometimes painfully, that the hill is steepest where the contour lines are closest together. The steepest direction of ascent or

descent is in the direction perpendicular to the contour lines or to the equipotential curves. If a direction s is chosen perpendicular to the contour lines, or s is in the direction of maximum slope, then along this direction the rate of change of potential energy dV/ds is a maximum. Along an equipotential curve in which V is constant the rate of change of the potential energy is zero; that is, there is no gravitational component of force along the horizontal equipotential curves. In any arbitrary direction, as for instance along the X axis, the rate of change of potential energy $\partial V/\partial x$ or the steepness of the hill in this direction is equal to the negative of the gravitational force in this direction.

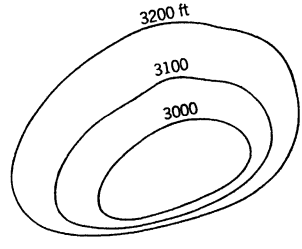


FIG. 3.5 A contour map showing places of equal elevation above sea level.

Frequently the potential energy per unit mass, called the gravitational potential, and the gravitational force per unit mass, called the gravitational intensity, are used rather than the potential energy of and gravitational force on a mass m. Gravitational intensity has the same dimensions as acceleration. If V' is the gravitational potential and g the gravitational intensity, then corresponding to Eqns. 3.6 and 3.7 we have

$$V' = -\int g\, dy$$

and

$$g = -\frac{\partial V'}{\partial y}$$

Largely for the sake of abbreviation a vector differential operator del ∇ is sometimes used. This operator is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

A conservative force $\mathbf{F}$ may be written in terms of this operator and the potential energy V. From Eqn. 3.10

$$\mathbf{F} = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z} = -\nabla V$$

This operator adds nothing to the analysis here, though it is used extensively in vector analysis.

3.5 Examples of Conservation of Energy

A. Let us consider a particle of mass m placed on a smooth inclined plane of height h and inclination θ, Fig 3.6. The potential energy of the mass m at any height y is

$$V = mgy$$

relative to its potential energy along the horizontal X axis. If the distance s is measured from the origin along the incline, then

$$y = s \sin \theta$$

and

$$V = mgs \sin \theta$$

The conservative force $\mathbf{F}_s$ acting on the mass in the direction of increasing s is

$$\mathbf{F}_s = -\frac{\partial V}{\partial s} = -\frac{\partial (mgs \sin \theta)}{\partial s} = -mg \sin \theta$$

Thus the force along the plane acting on the mass m is $mg \sin \theta$ in a direction down the plane. This same result was previously obtained by resolving the vertical force mg into components along and at right angles to the plane.

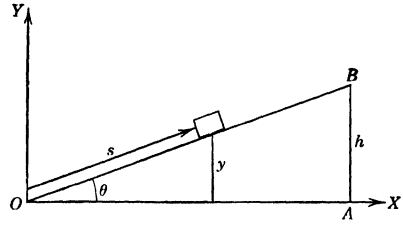


FIG. 3.6 Conservation of mechanical energy on a smooth inclined plane.

Since we have assumed that the plane is smooth, i.e., there is no force of friction, the system in Fig. 3.6 is a conservative one and the sum of the potential and kinetic energies of the mass as it moves down the plane is a constant. If the particle starts from rest at the top of the plane, its energy in this position is all potential and has a value of mgh. This potential energy is equal and opposite to the work done by the force F_s in moving the particle down the plane a distance BO. The work done is

$$F_s \frac{h}{\sin \theta} = mg \sin \theta \frac{h}{\sin \theta} = mgh$$

If v is the velocity of the particle at the bottom of the plane, then by the law of conservation of energy it follows that

$$\tfrac{1}{2}mv^2 = mgh$$

or

$$v^2 = 2gh$$

This is the same result as that obtained for a particle starting from rest and falling through a vertical height h with a constant accelceration g It could also be obtained by considering the motion of a particle down the plane where the acceleration is $g \sin \theta$ and the distance traveled is $h/\sin \theta$

B A particle weighing 2 lbw is acted on by a force F_y measured in pounds weight according to the equation

$$F_y = 5 - y - 2y^2$$

where y is measured in feet as shown in Fig. 3.7. The problem is to find the work done by the force when the particle moves from $y = 0$ to $y = 2$ ft, also to find the velocity of the particle at $y = 2$ ft, assuming that the velocity of the particle is zero at $y = 0$ ft The work done by the force on the particle in moving it from $y = 0$ to $y = 2$ ft is

$$W = \int_0^2 F_y \, dy = \int_0^2 (5 - y - 2y^2) \, dy$$

$$= \left[5y - \frac{y^2}{2} - 2\frac{y^3}{3} \right]_0^2$$

$$= 10 - 2 - 5\,33 = 2\,67 \text{ ft lbw}$$

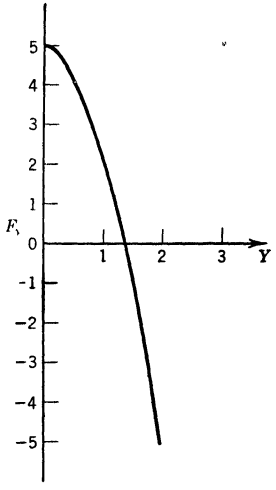


FIG. 3.7 Variation of force F_y with distance y.

It should be noted that the force F_y decreases as y increases from zero on, that F_y is zero at $y = 1.35$ and is negative thereafter. Thus the particle is repelled from the origin up to $y = 1.35$ and is attracted towards the origin thereafter. Though the force on the particle is zero at $y = 1.35$ ft, its velocity at this position is directed away from the origin since the total work done on the particle is positive from $y = 0$ to $y = 1.35$. From the expression for the work given above it can readily be shown that the total work done by the force on the particle is zero from $y = 0$ to $y = 2.39$ ft Since the particle is assumed to start from rest at $y = 0$, it follows that the particle is again at rest at $y = 2.39$ ft. The particle will not go beyond the point $y = 2\,39$ but will be attracted towards the origin and will subsequently oscillate about the equilibrium point at which $F_y = 0$, that is, the point $y = 1\,35$ ft

To complete our problem of determining the velocity of the particle at $y = 2$ ft we have for the work done on the particle from $y = 0$ to $y = 2$ ft a value of 2 67 ft lbw Since the velocity and kinetic energy of the particle is zero at $y = 0$, it follows that the kinetic energy at $y = 2$ ft is 2.67 ft lbw. Thus

$$\frac{1}{2}\frac{w}{g} v^2 = 2.67$$

or

$$v^2 = 32 \times 2.67$$

The velocity of the particle at $y = 2$ ft is approximately

$$v = 9.24 \text{ ft per sec}$$

As an alternate manner of deriving this velocity, the acceleration and then the velocity may be expressed as a function of y. The acceleration y of the particle is given by

$$F_y = \frac{w}{g} y = 5 - y - 2y^2$$

From the identity $(dy/dt)dt = dy$ and multiplying the left-hand side by $y\,dt$ and the right-hand side by dy, we have

$$\frac{w}{g} yy\,dt = (5 - y - 2y^2)\,dy$$

or

$$\frac{w}{2g} \frac{d(y^2)}{dt}\,dt = (5 - y - 2y^2)\,dy$$

Hence by integration we obtain for the kinetic energy of the particle

$$\frac{w}{2g} \dot{y}^2 = 5y - \frac{y^2}{2} - \frac{2y^3}{3} + C$$

where the constant of integration C is determined from the initial conditions, $\dot{y} = 0$ when $y = 0$, giving $C = 0$. The velocity y at $y = 2$ ft is then approximately 9.24 ft per sec as obtained above.

3.6 An Elastic Collision in the Center of Mass Coordinate System

Elastic collision theory is of importance in nuclear reactors in which fast neutrons are slowed down by colliding with stationary nuclei. In the first operating chain reactor of December 1942 the uranium that underwent fission was embedded in blocks of graphite. When fission takes place, neutrons of high energy are emitted, and these have to be slowed down or moderated in order to cause further fission. This slowing down is accomplished by the elastic collisions between the neutrons and the carbon nuclei. As may be shown by working problem 3 at the end of this chapter there is conservation of kinetic energy at an elastic collision. The fractional loss of kinetic energy of the neutron at each collision can be readily calculated from the laws of conservation of momentum and energy. However, the analysis is simplified if it is calculated with respect to a coordinate system attached to the center of the mass of the colliding particles. In order to illustrate the procedure we shall essentially repeat some of the material given in section 2.7.

Up to the present we have employed a coordinate system which is at rest with respect to the moving particles or one in which the target

nucleus is at rest before the collision. This system is frequently called the laboratory or L system inasmuch as this is the system in which all measurements are made. In contrast to this let us consider a collision in the center of mass or C system.* Suppose that a neutron of mass m moving to the right with a velocity v_0 collides with a stationary nucleus of mass M. After the collision the neutron and nucleus are scattered as shown in Fig. 3.8*a* for the laboratory system. In the C

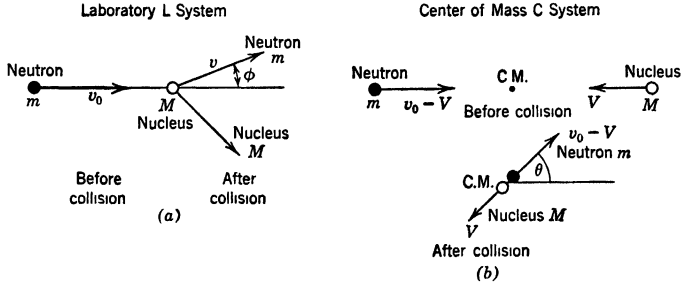


FIG. 3.8 Collision of a neutron with a nucleus in the laboratory and center of mass systems.

system an observer attached to the center of mass would see the collision, as shown in Fig. 3.8*b*. From the point of this C coordinate system the center of mass is at rest. In the L system the velocity V of the center of mass to the right is given from the law of conservation of momentum as

$$(M + m)V = mv_0 \qquad \text{or} \qquad V = \frac{mv_0}{(M + m)}$$

In the C system in which the center of mass is considered to be at rest the neutron is moving to the right with a velocity of

$$v_0 - V = \frac{Mv_0}{(M + m)}$$

and the nucleus is moving to the left with a velocity of

$$V = \frac{mv_0}{(M + m)}$$

*If you are not familiar with the center of mass, you will find a discussion in section 6.2.

In the C system the total momentum before collision is zero as is readily seen from the above equations, and from the law of conservation of momentum it is zero after the collision. Since there is conservation of kinetic energy at the elastic collision, the kinetic energy before impact must be equal to the kinetic energy after impact. This must mean that in the C system the speeds of the neutron and the nucleus must be unchanged at the collision, for a change in speeds would imply a change in the total kinetic energy. In the C system

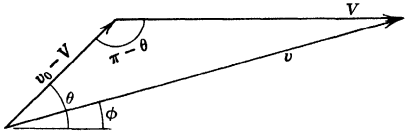


FIG. 3.9 The velocity v of the neutron after the collision in the L system determined from the velocity $v_0 - V$ in the C system and the velocity V of the C.M.

after the collision the two particles go off in opposite directions with their speeds unchanged, or the total effect of the collision is to change the direction but not the magnitudes of the velocities as shown in Fig. 3.8*b*. This is not true in the L system, in which both the magnitude and the direction of the velocities are changed. Since experimental measurements are made in the L system, it is necessary to change the velocities from the C to the L system. This is done by the vector velocity triangle, Fig. 3.9. The velocity v of the neutron after the collision is obtained in the L system by adding the velocity $(v_0 - V)$ in the C system to the velocity V with which the center of mass moves in the L system relative to the C system. From Fig 3.9 and the law of cosines and substituting for V and $(V - v_0)$, we obtain

$$v^2 = (v_0 - V)^2 + V^2 + 2V(V - v_0)\cos\theta$$
$$= \frac{{v_0}^2 M^2}{(M+m)^2} + \frac{{v_0}^2 m^2}{(M+m)^2} + \frac{2{v_0}^2 mM\cos\theta}{(M+m)^2}$$

The ratio of the kinetic energy E of the neutron after the collision to its kinetic energy E_0 before collision is given in the laboratory system as

$$\frac{E}{E_0} = \frac{mv^2/2}{m{v_0}^2/2} = \frac{M^2 + m^2 + 2mM\cos\theta}{(M+m)^2}$$

It follows that, the larger the mass of the nucleus, the smaller the amount of energy which the neutron loses on impact.

3.7 General Law of Conservation of Energy

In the foregoing examples we have neglected the force of friction though in practice it can never be entirely eliminated. Let us consider the first example dealing with a particle sliding down an inclined plane. If we now include friction, then with a coefficient of friction μ between the particle and the plane the force down the plane would be $(mg \sin \theta - \mu mg \cos \theta)$. In this case the kinetic energy of the particle at the bottom of the plane would not equal the potential energy at the top of the plane since some work would be done against the force of friction as the particle slides down the plane. The system is no longer a conservative one since the work done in taking the mass around a closed path, as for instance from the top to the bottom and back to the top of the plane, is not zero. When the mass moves up or down the plane, work is done against friction and this goes into an equivalent amount of heat energy.

About the middle of the last century Joule, an English physicist, carried out a large number of experiments in which the heat and the work against the force of friction were carefully measured. These experiments showed that 4.187 joules of work done against friction would heat 1 gm of water 1°C, that is, would produce 1 calorie of heat. It was not until after quite a controversy that heat became recognized as another form of energy. However, the results of the experiments of Joule and others could not be denied, and it was on the basis of these experiments that the general law of conservation of energy was enunciated. According to this law the total energy in a closed or isolated system is a constant. If the isolated system taken is our universe, it follows that the total energy of it is a constant independent of time. Although energy is continually changing from one form to another, the total energy of the universe must remain constant.

Since the time of Joule many other forms of energy have been recognized. Possibly the most important discovery in our time is that matter itself is a form of energy. This discovery was made by means of theoretical reasoning by Einstein who showed that any piece of matter of mass m grams has an energy equivalent to E ergs according to the relation

$$E = mc^2 \tag{3.11}$$

where c is the velocity of light, approximately 3×10^{10} cm per sec. Thus each gram of matter is equivalent to approximately 9×10^{20} ergs of energy. The practical possibility of changing matter into energy did not come until the early thirties when research with nuclear accelerators was started. Until the advent of the atomic bomb the amount of matter ever transformed into energy was exceedingly minute.

All the various forms in which energy can exist are included in the general law of conservation of energy, making this a law of universal application. This law in a sense disposes of the law of conservation of mass inasmuch as the latter is included in the conservation of energy law. However, classical physics deals with such relatively small amounts of energy that their mass equivalents are negligible, so we shall continue to use the law of conservation of mass.

3.8 Kinetic Energy in Relativity Theory

Such a section as this does not really belong in any discussion of classical physics, but having already included some discussion of the relativity theory it seems appropriate to give the equation for kinetic energy. According to the relativity theory, if a body has a mass m_0 when at rest, then when it is moving with a velocity v its mass relative to an observer at rest is given by

$$m_v = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

and its kinetic energy is

$$\text{K.E.} = m_0c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) \tag{3.12}$$

where c is again the velocity of light. If the velocity of the mass is small compared to the velocity of light, then the relativistic equation for kinetic energy may be written

$$\begin{aligned}\text{K.E.} &= m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] \\ &= m_0c^2\left[1 + \frac{v^2}{2c^2} \cdots - 1\right] \\ &= \tfrac{1}{2}m_0v^2\end{aligned}$$

when terms in v^4/c^4 and higher powers are neglected. Thus we are justified in using the expression $m_0v^2/2$ for the kinetic energy of bodies such as moving automobiles or airplanes. The Newtonian or classical theory is the limit of the relativistic theory when the velocities of the moving bodies are small compared to the velocity of light.

As an example let us consider how the relativity theory affects the mass and kinetic energy of an airplane traveling at the very large speed of 1000 mph. The ratio of the velocity of the plane to the velocity of light (186,000 mi per sec)

$$\frac{v}{c} = \frac{1}{186} \times \frac{1}{3600} = 1.493 \times 10^{-6}$$

Hence

$$\frac{v^2}{c^2} = 2.229 \times 10^{-12}$$

and

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + 1.115 \times 10^{-12}$$

Thus the increase in mass is quite negligible, and the relativistically kinetic energy is negligibly different from that obtained by the classical theory.

3.9 Power

Frequently one is interested not only in the amount of work done by a force but also in the time taken to do this work. The time rate of doing work is called power. If an amount of work ΔW is done by a force in a time Δt, the average power developed $\bar{P}$ is

$$\bar{P} = \frac{\Delta W}{\Delta t}$$

The power P developed at any instant is given by the limit of this expression as Δt approaches zero or

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} = \dot{W} \tag{3.13}$$

If a force $\mathbf{F}$ moves a particle through a displacement $d\mathbf{r}$ in a time dt, the work done is

$$dW = F \cdot d\mathbf{r}$$

and the instantaneous power developed by the force is

$$\frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{3.14}$$

where $\mathbf{v}$ is the instantaneous velocity of the particle.

The dimensions of power in the absolute system of units are $[ML^2T^{-3}]$ and in the gravitational system are $[FLT^{-1}]$. In the cgs system the unit of power is the erg per second, which is very small for practical purposes, so the unit of the mks system is used instead. This is the joule per second, also called a *watt*. Energy and work are frequently measured in units of power multiplied by time, as for example in kilowatt hours, kwh. In the British engineering system of units the unit of power is the foot pound weight per second. Another unit is the horsepower which is equal to 550 ft lbw per sec or to 746 watts.

An example of energy and power. A train is traveling at a constant speed of 60 mph and is picking up water from a trough between the rails. If it picks up

15,000 lb of water in 400 yd, let us determine the extra force and horsepower required.

The speed of the engine is 60 mph or 88 ft per sec. It goes 400 yd in $(400 \times 3)/88$ sec so that the time rate of picking up water is $(15{,}000 \times 88)/(400 \times 3) = 1100$ lbw per sec The force F which the engine must exert to pick up this water is given by Newton's second law as

$$F = \frac{d(mv)}{dt} = v\frac{dm}{dt} = \frac{v\,dw}{g\,dt}$$

$$= \tfrac{88}{32} \times 1100 = 3025 \text{ lbw}$$

since the velocity v is constant. The additional horsepower which the engine must develop to maintain its constant speed and pick up the water is

$$\frac{Fv}{550} = \frac{3025 \times 88}{550} = 484 \text{ hp}$$

Now it might be thought that the additional force necessary could be found from the relationship between the work done by the force F and the increase in kinetic energy of the water An amount of kinetic energy

$$\text{K.E.} = \frac{1}{2}\frac{w}{g}v^2 = \frac{1}{2} \times \frac{15000}{32} \times 88 \times 88 = 1.815 \times 10^6 \text{ ft lbw}$$

is picked up in a distance of 400 yd, so the force required for this is

$$F = \frac{1.815 \times 10^6}{1200} = 1512.5 \text{ lbw}$$

This is exactly half the value obtained by using the rate of change of momentum. The discrepancy lies in the fact that the engine has done more work than that required to give the water kinetic energy. Some energy goes into heat and, as illustrated in problem 21, is exactly equal to the kinetic energy received Thus the total amount of energy received by the water in the 400 yd is double that given by the kinetic energy, or 3.63×10^6 ft lbw. With this value for the energy the force comes out correctly. For the situations in which a mass in motion captures another object, or whenever one mass is added to or subtracted from another, the force exerted cannot be obtained directly from the relationship between the work done and the increase in kinetic energy.

PROBLEMS

1. A gun having a mass of 5 kg fires a bullet whose mass is 20 gm with a velocity of 30,000 cm per sec.

a. Find the kinetic energy of the gun immediately after firing the bullet and compare this with the initial kinetic energy of the bullet. Discuss the energy relationships in this problem.

b. If the bullet goes down the barrel in 0.01 sec, find the average force exerted on the bullet.

2. A bullet whose mass is 10 gm and which has a velocity of 30,000 cm per sec is fired into a block of wood whose mass is 10 kg and which is suspended so as to swing as a pendulum. The bullet penetrates 3 cm into the block of wood before coming to rest.

a. Find the vertical height through which the block and bullet are raised.

b. Find the amount of heat produced in the block and bullet.

c. Find the average resistance force exerted by the block on the bullet.

3. Two spheres A and B moving in the same direction impact along their line of centers. The mass of A is 10 gm, and its initial velocity is 100 cm per sec. The mass of B is 20 gm, and its initial velocity is 40 cm per sec. Find the loss of energy at impact for coefficients of restitution of (*a*) 0; (*b*) 1.0; (*c*) 0.8.

4. A body whose weight is 10 lbw is pulled up a smooth inclined plane whose angle of inclination is 30° and whose length is 1 meter by a force of 8 lb making an angle of 60° with the horizontal. If the body starts from rest at the bottom of the plane, find its kinetic energy and velocity at the top of the plane. Show that the work done by the force in pulling the body up the plane is equal to the increase in mechanical energy.

5. If the plane in problem 4 is rough with a coefficient of friction of 0.15, find the kinetic energy and velocity of the body at the top of the plane. Show that the difference between the work done by the force and the increase in mechanical energy is equal to the work done against the frictional force.

6. A body having a weight of 10 lb is projected from the bottom of an inclined plane, having an angle of inclination of 30°, with an initial velocity of 30 ft per sec. If the coefficient of friction between the body and the plane is 0.20, find:

a. The distance the body moves up the plane before coming to rest.

b. The work done against friction in bringing the body to rest. Show that this is equal to the difference between the initial kinetic energy and the final potential energy.

c. The velocity with which the body reaches the bottom of the plane on sliding down. Check this answer by finding the kinetic energy at the bottom of the plane using the data obtained in part *b*.

7. An automobile whose weight is 3000 lb is traveling at 60 mph. It can be stopped in a distance of 100 ft after the brakes are applied.

a. Find the value of the braking force.

b. Find the distance in which this automobile would be stopped when going at 30 mph.

c. Find the distance in which this automobile would be slowed from 60 to 30 mph.

d. Compare the time of stopping from 60 mph with that from 30 mph.

8. A steel ball falls from a height h_0 and strikes a large steel block. If it rebounds to a height h_1, find the coefficient of restitution for the block and ball in terms of h_0 and h_1. Find the height h_2 of the second rebound and the height h_n of the nth rebound in terms of h_1 and h_0.

9. A mass of 10 gm is hung on the end of a light inextensible cord 100 cm in length so as to form a simple pendulum. The cord is pulled aside so that it makes an angle of 30° with the vertical.

a. Find the potential energy of the system relative to the lowest point of its swing.

b. If the system is released, find the velocity of the mass at the lowest point of swing.

c. Find the tension in the cord at the lowest point of swing.

d. Find the kinetic and potential energies of the system when the cord makes an angle of 20° with the vertical.

10. Assuming that the tangential components of velocity remain unaltered at an oblique impact of a sphere on a plane, show that the law of reflection for the

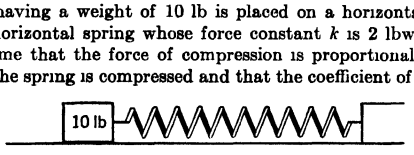
sphere on the plane is $e \cot \theta = \cot \phi$ where e is the coefficient of restitution between the sphere and the plane, and θ and ϕ are the angles of incidence and reflection respectively.

11. A freight car weighing 8 tons is moving with a speed of 30 mph when it strikes a spring bumper and is brought to rest by the force of compression of the spring. If the force of compression F is proportional to the distance x through which the spring is compressed, or $F = kx$ where k is a constant equal to 500,000 lbw per ft, then:

a. Show that the work done in compressing the spring is $kx^2/2$.

b. Find the distance through which the spring is compressed in bringing the car to rest.

12. A block having a weight of 10 lb is placed on a horizontal surface and is attached to a horizontal spring whose force constant k is 2 lbw per in. of compression. Assume that the force of compression is proportional to the distance through which the spring is compressed and that the coefficient of friction between



PROBLEM 12

the block and the horizontal surface is 0 20. A bullet whose weight is 0.125 lb and whose velocity is 1000 ft per sec is fired into the block and comes to rest in it.

a. Find the distance the block moves before it comes to rest.

b. Assuming that the force constant for extension is the same as that for compression, find the second position of rest.

c. Describe qualitatively the subsequent motion of the block.

13. Find the equivalent number of:

a. Joules in a kilowatt-hour.

b. Calories in a kilowatt-hour.

c. Watts in a horsepower.

d. Joules in a horsepower-hour.

14. A block whose mass is 50 gm placed on a horizontal frictionless surface moves under the action of a horizontal force given in dynes as

$$F = -10^3x$$

where x is measured in centimeters. Originally the block is at rest at a distance of 5 cm from the origin of the X axis Find:

a. The work done by the force when the block moves to the origin.

b. The kinetic energy and speed of the block at the origin.

c. The kinetic and potential energies of the block at $x = 2$ cm.

15. A mass of 100 gm is placed on the end of a spring whose restoring force is proportional to the extension. When 100 gm is placed on the lower end of the spring, which is rigidly held at its upper end, the extension is 2 cm. Initially the 100-gm mass is pulled downward 5 cm below the equilibrium position.

a. Find the potential energy of the system

b. Find the kinetic energy of the mass as it passes the equilibrium position.

c. At what displacement is the speed again zero?

16. A block whose mass is 100 gm is placed on a frictionless horizontal surface and is acted on by a horizontal force F in dynes of

$$F = -10^3x - 10x^3$$

where x is measured in centimeters. The block is originally at rest at the origin.

a. Find the work done against the force when the block is moved to $x = 5$ cm If the block is released from rest at $x = 5$ cm and moves toward the origin, find·

b. The kinetic and potential energies at $x = 2$ cm.

c. The potential energy at any position x and show that the force at any point x is given by the space derivative of the potential energy at that point.

17. An electrically charged body whose mass is 5 gm is attracted by another charged body with a force of

$$F = -\frac{10^4}{r^2} \text{ dynes}$$

where r is measured in centimeters.

a. Find the work done against the force when the charged body is moved from $r = 2$ to $r = 5$ cm.

b. What is the shape of the equipotential surfaces about the charged body?

c. If the potential energy of the 5-gm mass is considered to be zero at an infinite distance, find its potential energy at $r = 10$ cm

18. An automobile can go up a hill whose inclination is 1 in 60 with a constant speed of 60 mph when the resistance force is $\frac{1}{30}$ of the weight of the car. If this weight is 3000 lb:

a. Find the horsepower exerted by the car.

b. If half of this horsepower is exerted when coming down the hill and the resistance force is the same as that going up, find the constant speed with which the automobile comes down the hill.

19. Find the horsepower developed by a fire engine which can project 200 lb of water per second with a nozzle speed of 88 ft per sec

20. A uniform flexible rope having a weight of 4 lb per ft is drawn from a heaped-up coil with a constant speed of 6 ft per sec along a frictionless floor. Find the force exerted and the horsepower developed.

21. A number of 1-gm masses are placed in a straight line at equal intervals of 2 cm apart on a frictionless horizontal table A 10-kg mass is pushed along the table so as to make inelastic impacts with each of the 1-gm masses, thus adding to its own mass. If the 10-kg mass is pushed along the table at a constant speed of 20 cm per sec, find:

a. The kinetic energy received by the 1-gm masses each second.

b. The mechanical energy lost at each impact and the energy lost per second owing to the inelastic impacts.

c. The force necessary to maintain the constant velocity both from the total work done per second and from the time rate of change of momentum.

22. *a.* Find the speed with which a 1000-hp engine can pull a train weighing 500 tons along a horizontal track if the resistance forces are constant at 15 lbw per ton.

b. Find the acceleration of the train when its speed is 20 mph.

c. Find the maximum speed with which the engine could pull the train up an incline of 1 in 200 against the same resistance forces

23. Find the energy equivalent of 1 microgram of mass. At the price of 3 cents per kilowatt-hour how much is this energy worth?

24. If the mass of a body at rest is 10 gm, find the mass of the body relative to an observer at rest when the body has a speed of

a. 3×10^9 cm per sec. *b.* 2.7×10^{10} cm per sec.

c. Compare the kinetic energy of a body moving with a speed of 3×10^9 cm per sec as given by classical mechanics with that obtained from relativistic mechanics.

25. Show that the maximum fractional loss in energy in an elastic collision between a neutron and a stationary nucleus occurs when there is a head-on collision, and find this fractional loss for a collision with a deuteron, mass 2, and a carbon nucleus, mass 12. Take the mass of the neutron as unity. Make these calculations in both the laboratory and the center of mass coordinate systems.

26. It appears that the scattering of neutrons by nuclei is spherically symmetric in the center of mass coordinate system for relatively light nuclei and neutron energies of a few thousand electron volts. This implies that all values of $\cos \theta$ in Fig. 3.9 are equally probable. Show that this also means approximately spherically symmetric scattering in the laboratory L system for nuclei in which M/m is large but not for hydrogen nuclei for which $M = m = 1$.

4 · NEWTON'S LAW OF GRAVITATION AND SOME OF ITS CONSEQUENCES

4.1 The Law of Gravitation

During the great plague in 1666 the English universities closed and Newton, who was then an undergraduate at Cambridge, returned to his home in the midlands of England. While sitting in the orchard he is said to have pondered on the problem of the fall of the apple. (Apples have played other important roles in the history of man.) Newton was interested in whether any common principle could explain the falling apple and the falling of the moon toward the earth as it revolves about the earth. It appears that Newton made some rough calculations, assuming that the earth attracted both the apple and the moon with a force proportional to the inverse square of the distance of the respective body from the center of the earth. These calculations can be repeated by solving problem 13 at the end of the chapter. Newton then extended this assumption into what has now become known as *Newton's law of gravitation*, which may be stated as follows:

Every particle in the universe attracts every other particle with a force which is proportional to the product of their masses and inversely proportional to the square of their distance apart.

As we shall see, one of the important contributions made by this law is in the interpretation of the motions in our solar system.

Mathematically the law of gravitation may be stated in the following manner. If m_1, m_2 are the masses of two particles separated by a distance r, the force of attraction F along the line joining the particles is

$$F \propto \frac{m_1 m_2}{r^2}$$

This equation may also be written

$$F = G\frac{m_1 m_2}{r^2} \tag{4.1}$$

where G, the constant of proportionality, is called the *constant of gravitation*.

The earliest experiments for determining G were made in the eighteenth century by the Reverend Mitchell and by Lord Cavendish. These experiments were repeated in the nineteenth century by Professors Poynting and Boys with improved apparatus. The present accepted value of G obtained by Dr. Heyl* at the National Bureau of standards in Washington is

$$G = 6.670 \times 10^{-8} \frac{\text{dyne cm}^2}{gm^2}$$

with a probable error of $\pm 0.005 \times 10^{-8}$.

Essentially all these determinations were carried out by the method indicated in Fig. 4.1. In this method a light rod has a mass m placed on each end. The rod and masses are suspended horizontally by a

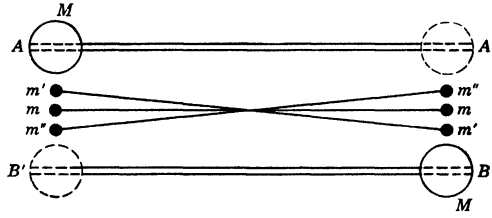


FIG. 4.1 Cavendish method for measuring the gravitational constant G.

vertical fine fiber. Close to each mass m is a large mass M. When the large masses are in the positions A and B, the small masses are attracted and take the equilibrium position at $m'm'$. When the large masses M are moved to the position $A'B'$, the rod and light masses move to a position indicated at $m''m''$. If the masses and their distances apart, the torsion constant of the fiber, and the angles of torsion are known, it is possible to calculate G. Such a calculation is called for in problem 23, Chapter 6. Since the force of attraction is very small, the fibers must have a very small torsion constant. Professor Boys made a significant improvement when he developed a method for drawing very fine quartz fibers.

In the statement of Newton's law of gravitation the masses are considered to be particles, and since the earth is anything but a particle from the point of view of a falling apple the question arises as to what is the appropriate distance between the earth and the apple. To solve this problem Newton is said to have invented the calculus. All this he did as an undergraduate while on vacation!

* P. R. Heyl, *Proc. Nat. Ac. Sci.*, **13**, 601 (1927).

4.2 The Gravitational Attraction between a Uniform Sphere and a Particle

We shall now prove that the earth attracts an external particle as though all the mass of the earth were concentrated in a point at its center. To do this we must first consider the attraction between a uniform spherical shell and an external particle, since a solid sphere may be thought of as built up from a large number of concentric thin shells.

Let R be the radius of a thin shell whose mass per unit area is σ. The problem is to find the force of gravitational attraction between this shell and a particle of mass m at P located at a distance l from the center of the shell, as shown in Fig. 4.2. In order to find this force we

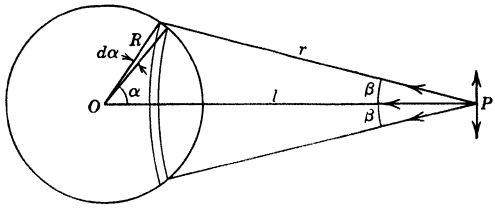


FIG. 4.2 Attraction of a uniform shell and an external particle.

first divide the shell into a number of strips formed by the intersection of the shell with spheres of radii r and $r + dr$. The particular strip shown in Fig. 4.2 has a width $R\,d\alpha$. The area of the strip is $2\pi R \sin\alpha\, R\,d\alpha$, and its mass is $\sigma 2\pi R^2 \sin\alpha\,d\alpha$. Every element on this strip is, to a first approximation, at the same distance r from the particle at P. The force of attraction between any element in this strip and the particle at P is along a straight line drawn between the element and P. Thus the vectors representing the forces at P form a cone with its vertex at P. Two of these vectors are shown in the plane of the paper. The forces at P can be resolved into components along and at right angles to PO. It is easily seen that the components at right angles to OP cancel and have a resultant of zero, whereas the components along PO add together. The resultant force between the mass m at P and the strip is along the line PO and has a magnitude of

$$dF = G\frac{(2\pi\sigma R^2 \sin\alpha\,d\alpha)}{r^2} m\cos\beta \qquad (4.2)$$

Since the values of r, α, and β are different for each strip, it is not possible to integrate Eqn. 4.2 directly to find the force between the shell and

the particle. We must eliminate some of the variables. From the cosine formula we have

$$r^2 = R^2 + l^2 - 2Rl \cos \alpha$$

and by differentiation

$$2r\, dr = 2Rl \sin \alpha\, d\alpha$$

since R and l are constants. Thus

$$\sin \alpha\, d\alpha = \frac{r\, dr}{lR}$$

Similarly

$$\cos \beta = \frac{r^2 + l^2 - R^2}{2rl}$$

Substituting for $\sin \alpha\, d\alpha$ and $\cos \beta$ in Eqn. 4.2, we have

$$dF = G \frac{\sigma \pi m R}{l^2} \left(1 + \frac{l^2 - R^2}{r^2}\right) dr$$

Since the total mass of the shell m_s is $4\pi R^2 \sigma$, the force dF may be given as

$$dF = G \frac{m_s m}{4Rl^2} \left(1 + \frac{l^2 - R^2}{r^2}\right) dr$$

This equation now contains only the variable r so that the resultant force F between the shell and the particle can be obtained by integrating this equation between the limits $(l - R)$ and $(l + R)$. By integration

$$\begin{aligned} F &= \frac{Gm_s m}{4Rl^2} \left[r - \frac{l^2 - R^2}{r} \right]_{l-R}^{l+R} \\ &= \frac{Gm_s m}{4Rl^2} \left[2R + \frac{2R(l^2 - R^2)}{l^2 - R^2} \right] \\ &= \frac{Gm_s m}{l^2} \end{aligned} \tag{4.3}$$

This is the value of the force of attraction between two particles of mass m_s and m respectively, separated by a distance l. Thus, as far as the attraction of the shell and an external particle is concerned, the shell behaves as though all its mass were concentrated at its center. Since a uniform solid sphere may be considered to be built up of an infinite number of concentric shells, it follows that a uniform solid sphere attracts an external particle as though all the mass of the sphere

were concentrated at a point at its center. If the earth is a uniform sphere or made up of uniform concentric shells, it attracts an external particle such as the apple as though all the mass of the earth were concentrated at a point at its center.

In problem 2 at the end of the chapter one is asked to show that the resultant force on a particle inside a closed shell is zero. There is no gravitational force inside a closed shell, and the gravitational potential within the shell is constant and equal to the value on its surface. This is analogous to a similar conclusion made in electrostatics, namely that the electric field inside a closed charged conductor is zero and the electric potential inside the shell is constant and equal to the value at its surface. These results are a consequence of the inverse square law which holds for point electric charges and for particles.

It follows from these results that the force on a particle within a solid sphere is due to the mass within the spherical portion of the sphere enclosed by the particle, since the portion of the sphere outside the particle exerts no force on the particle.

4.3 Gauss's Law

Another manner of stating the inverse square law, which is useful in solving problems involving symmetrical figures, was first given by the German scientist Gauss and is called Gauss's law. This law may be stated in the following manner: The total outward gravitational flux through any closed surface is equal to $-4\pi G$ times the total mass enclosed by the surface.

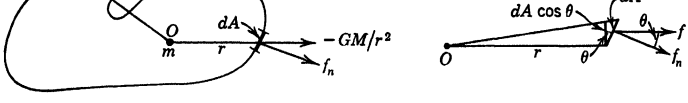


FIG. 4.3 Gauss's law for a closed surface about a mass m.

If f_n is the normal component of the gravitational force on a unit mass at an element of area dA, then $f_n\, dA$ is defined as the gravitational flux through dA, and Gauss's law may be written

$$\int_{\substack{\text{closed}\\ \text{surface}}} f_n\, dA = -4\pi GM$$

where M is the total mass enclosed within the surface. In order to prove this law let us consider a mass m at O, Fig. 4.3, with a closed

surface drawn around it. If dA is a small element of area in the surface at a distance r from O, then the gravitational force exerted on a unit mass at dA by the mass m is Gm/r^2. This force is directed along r from dA to O, and the normal component of this force is f_n in an outward direction where

$$f_n = -\frac{Gm \cos \theta}{r^2}$$

The gravitational flux through dA is

$$f_n \, dA = -\frac{Gm \cos \theta \, dA}{r^2}$$

But $dA \cos \theta$ is the component of the area dA perpendicular to the radius r, and $(dA \cos \theta)/r^2$ is the element of solid angle $d\Omega$ subtended by dA at O. Hence

$$f_n \, dA = -Gm \, d\Omega$$

and integrating this expression over any closed surface gives

$$\int_{\substack{\text{closed} \\ \text{surface}}} f_n \, dA = -Gm \int_{\substack{\text{closed} \\ \text{surface}}} d\Omega = -4\pi Gm$$

If the surface drawn about the mass has re-entrant portions as shown in the figure, then there must always be an odd number of times in which the lines from elementary cones drawn from O to the outside must cut the surface so that the expression given holds for any surface enclosing a mass. When the surface encloses a number of separate masses m_1, m_2 etc., whose total mass is M, then Gauss's law becomes

$$\int_{\substack{\text{closed} \\ \text{surface}}} f_n \, dA = -4\pi G(m_1 + m_2 \cdots) = -4\pi GM$$

As an example let us consider the gravitational force on a unit mass placed outside a homogeneous sphere of mass M, Fig. 4.4. Since there is symmetry, we shall choose as the closed surface a sphere having the same center as the homogeneous mass. This closed surface shown dotted in the figure has a radius R larger than that of the mass. Since f_n is everywhere the same on the surface, the total gravitational flux over the closed surface $\int f_n \, dA$ is in this case $4\pi R^2 f_n$. Thus from Gauss's law we have

$$4\pi R^2 f_n = -4\pi GM \qquad \text{or} \qquad f_n = \frac{-GM}{R^2}$$

Hence the gravitational force on a unit mass at a distance R from the

center of a homogeneous sphere of mass M is $-GM/R^2$, where the negative sign indicates that there is attraction between the mass M and the unit mass. This is essentially the result given in Eqn. 4.3. It

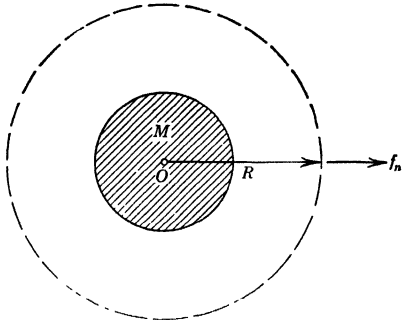


FIG. 4.4 Gravitational force at a distance R from a homogeneous sphere of mass M.

should be stated that Gauss's law is of great importance in electrostatics where the inverse square law holds between point charges.

4.4 The Acceleration Due to Gravity and the Law of Gravitation

The force the earth exerts on a mass m near its surface is given by the law of gravitation as

$$F = G\frac{Mm}{R^2}$$

where M is the mass and R the radius of the earth. This force F acting on the mass m is its weight mg, where g is the acceleration due to gravity. Hence

$$mg = G\frac{Mm}{R^2} \tag{4.4}$$

or

$$g = G\frac{M}{R^2} \tag{4.5}$$

If g and R are known from independent experiments, the measurement of G, the gravitational constant, provides a means of obtaining the mass of the earth, which is about 6.0×10^{27} gm or 6.7×10^{21} tons.

4.5 The Variation of the Acceleration Due to Gravity with Elevation

From the previous discussion it follows that the acceleration due to gravity at any point varies as the inverse square of the distance from the center of the earth to the point. The change in this acceleration is negligibly small for objects falling from points near the earth, but this is not so where the distance of fall is an appreciable fraction of the radius of the earth, 4000 miles.

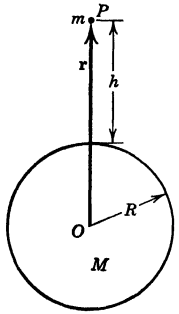


FIG. 4.5 Variation of gravity with elevation.

Let O be the center of the earth of radius R and mass M, Fig. 4.5. A particle of mass m is placed at P at a distance r from the center of the earth. In order to locate the point P let O be the origin of the vector $\mathbf{r}$ which is equal to OP. The force on the mass m is

$$\mathbf{F} = -G\frac{Mm}{r^2}\mathbf{r}_1$$

where the negative sign indicates that the force $\mathbf{F}$ is in the opposite direction to the vector $\mathbf{r}$ or the unit vector $\mathbf{r}_1$. By Newton's second law the equation of motion of the particle at P is

$$m\ddot{r}\mathbf{r}_1 = -G\frac{Mm}{r^2}\mathbf{r}_1$$

or

$$\ddot{r} = -G\frac{M}{r^2} \tag{4.6}$$

To integrate Eqn. 4.6 or to change from an acceleration to a velocity we use the identity

$$\dot{r}\,dt = dr$$

Thus

$$\ddot{r}\dot{r}\,dt = -G\frac{M}{r^2}\,dr$$

or

$$\frac{1}{2}\frac{d}{dt}(\dot{r}^2)\,dt = \frac{1}{2}d(\dot{r}^2) = GM\,d\left(\frac{1}{r}\right) \tag{4.7}$$

For the initial conditions suppose that the mass m starts from rest at a height h above the surface of the earth, or, at $t = 0$, $\dot{r} = 0$ and $r = R + h$. Then the velocity v with which the mass strikes the ground is given by

$$\frac{1}{2}\left[\dot{r}^2\right]_0^v = GM\left[\frac{1}{r}\right]_{R+h}^R$$

Thus

$$v^2 = 2GM\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{2GMh}{Rh + R^2} \tag{4.8}$$

If the particle starts at a very large distance from the earth so that h is much larger than R, then the velocity v_∞ with which it strikes the earth when falling from an infinite distance is

$$v_\infty = \sqrt{\frac{2GM}{R}} \tag{4.9}$$

This is likewise the vertical velocity a particle must have if it is to permanently escape from the earth's gravitational force. This velocity of escape from the earth, v_∞, is about 7 mi per sec.

If the body falls from a height h which is small relative to the radius of the earth, its velocity v on reaching the earth is given by $v^2 = 2gh$. This equation assumes a constant gravitational acceleration g. For the condition that h is very small compared to R, the term Rh in Eqn. 4.8 is negligible compared to R^2 and the velocity is given by

$$v^2 = \frac{2GMh}{R^2}$$

Using Eqn. 4.5 where $g = GM/R^2$, it follows that

$$v^2 = 2gh$$

as is given by the simpler analysis in Chapter 1.

We shall now determine the time a body takes to fall from a large elevation h to the earth. For convenience let $R + h = R_1$. Then the velocity $\dot{r}$ at any distance r from the center of the earth is obtained from Eqn. 4.8.

$$\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{R_1}\right)$$

Thus

$$\frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R_1}} \tag{4.10}$$

The square root with the minus sign has been chosen because the velocity dr/dt is positive in the upward direction of increasing r whereas the right-hand side of the equation decreases as r increases.

In order to integrate Eqn. 4.10 let us change the variable from r to θ by making the substitution

$$r = R_1 \cos^2 \theta$$

Then

$$dr = -2R_1 \cos \theta \sin \theta \, d\theta$$

Substituting these values in Eqn. 4.10 we have

$$\sqrt{2GM}\, dt = \sqrt{\frac{{R_1}^2 \cos^2 \theta}{R_1(1 - \cos^2 \theta)}}\, 2R_1 \cos \theta \sin \theta \, d\theta$$

or

$$\sqrt{2GM}\, dt = 2{R_1}^{3/2} \cos^2 \theta \, d\theta$$

$$= {R_1}^{3/2} (1 + \cos 2\theta) \, d\theta$$

Integrating

$$\sqrt{2GM}\, t = {R_1}^{3/2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= {R_1}^{3/2} \left[\cos^{-1} \sqrt{\frac{r}{R_1}} + \sqrt{\frac{r(R_1 - r)}{{R_1}^2}} \right] + C$$

If we again assume the same initial conditions as above, namely that, at $t = 0$, $r = R_1$, then the constant of integration C is zero. The time T taken by the body to reach the earth is found by putting $r = R$, where R is the radius of the earth.

$$T = \sqrt{\frac{R_1}{2GM}} \left[R_1 \cos^{-1} \sqrt{\frac{R}{R_1}} + \sqrt{RR_1 - R^2} \right] \qquad (4.11)$$

4.6 Gravitational Potential Due to the Earth

The difference in potential energy between two points is the negative of the work done by the conservative force F_r in moving a body between the two points:

$$F_r \, dr = -dV'$$

or

$$F_r = -\frac{dV'}{dr}$$

The conservative force of gravity is downward whereas the potential energy V' increases upward. Hence the negative sign. Rather than deal with a change in potential energy of some mass m it is common to use the change in potential energy per unit mass called the change in potential. If **f** is the force per unit mass, or the gravitational accelera-

tion, and dV the change in potential in a distance $d\mathbf{r}$, then

$$\mathbf{f} \cdot d\mathbf{r} = -dV$$

At a distance r from the center of the earth, Fig. 4.5, the force on unit mass $\mathbf{f}$ or the intensity of the gravitational field is

$$\mathbf{f} = -\frac{GM}{r^2}\mathbf{r}_1$$

The negative sign is due to the fact that $\mathbf{f}$ is positive downward whereas the vector $\mathbf{r}_1$ is positive in an upward direction. When a unit mass is moved a distance $d\mathbf{r}$ upward, the potential increases by an amount dV in such a way that

$$\mathbf{f} \cdot d\mathbf{r} = -\frac{GM}{r^2}\mathbf{r}_1 \cdot d\mathbf{r} = -dV$$

Since $\mathbf{r}_1$ and $d\mathbf{r}$ are in the same direction, the dot product is equal to the scalar dr and

$$dV = \frac{GM}{r^2}dr$$

If the potential is V_0 at a distance r_0 from the center of the earth and V at a distance r, then by integration

$$\left[V\right]_{V_0}^{V} = \left[-\frac{GM}{r}\right]_{r_0}^{r}$$

or

$$V - V_0 = GM\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

If the gravitational potential is taken as zero at an infinite distance, or $V_0 = 0$ when $r_0 = \infty$, the potential at any distance r from the center of the earth is

$$V = -\frac{GM}{r} \tag{4.12}$$

The presence of the negative sign sometimes gives trouble in Eqn. 4.12. This need not be so if we bear in mind the definition of potential difference. Since the gravitational force is directed toward the earth, the potential increases away from the earth or upward. If the potential is taken as zero at an infinite distance, then it must become increasingly negative in a direction toward the earth. The smaller the value of r, the smaller, that is to say the more negative, the potential becomes.

The intensity of the earth's gravitational field $\mathbf{f}$ or the gravitational acceleration at a distance r from the center of the earth in the direction of increasing r is found to be

$$\mathbf{f} = -\frac{dV}{dr}\mathbf{r}_1 = -\frac{d}{dr}\left(\frac{GM}{r}\right)\mathbf{r}_1 = -\frac{GM}{r^2}\mathbf{r}_1$$

This is to be expected, since we started with this expression for gravitational intensity.

4.7 Kepler's Laws

During the seventeenth century there was considerable interplay between the sciences of astronomy, mathematics, and physics. Late in the sixteenth century a Polish monk, Copernicus, had accounted for the motion of the planets and stars by assuming a central sun with the earth and other planets revolving around it. This was called the heliocentric or Copernican theory in contrast to the earlier geocentric or Ptolemaic theory which assumed a central stationary earth with the sun and planets revolving around it. Ptolemy's theory had been accepted for about fifteen centuries, but during this time there had developed an increasing number of discrepancies between the observations and the theory.

On the observational side Tycho Brahe, the last great astronomer to make observations without the aid of a telescope, had carefully collected data on the position of the planet Mars. This data was given to his student Kepler, who worked nearly twenty years before he obtained any satisfactory explanation of the observations. His results are summarized in what have become known as Kepler's three laws. They are:

1. *The law of elliptical orbits: The planets move in ellipses about the sun which is located at one of the foci.*
2. *The law of areas: The line joining any planet to the sun sweeps out equal areas in equal intervals of time.*
3. *The harmonic law: The square of the period of any planet about the sun is proportional to the cube of its mean distance from the sun.*

These are empirical laws inasmuch as they were derived directly from observational data without any interpretation by physical theory. Newton was able to show that Kepler's laws could be derived from the laws of motion and the law of gravitation. This was a great triumph for the Newtonian theory and had much to do with the subsequent rapid increase in knowledge in the physical sciences. We shall now repeat Newton's analysis, though not in the form given in *The Principia.*

4.8 Motion as Governed by Newton's Law of Gravitation

Let us consider the motion of a particle of small mass m which is attracted by a particle of large mass M. The force of gravitational attraction between these masses is along the line joining them, so the resulting motion is often called *motion under a central force.* The acceleration of the large mass M will be much smaller than that of the small mass m, so that without making too much error we may consider

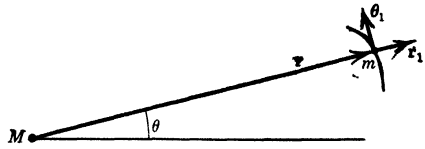


FIG. 4.6 Motion of a small mass m under the gravitational field of a large mass M.

M to be at rest with m moving about it. Any motion under a central force takes place in a plane which we shall here assume to be the plane of this page. Suppose the origin of the coordinates to be at M and the position of m to be given by the vector $\mathbf{r}$ joining M to m and making an angle θ with some fixed direction, Fig. 4.6. The coordinates r, θ are the polar coordinates of m with respect to M.

(By Newton's law of gravitation the force $\mathbf{F}$ exerted by M on m is

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}_1$$

where $\mathbf{r}_1$ is the unit vector in the direction of $\mathbf{r}$, namely from M to m. The general expression for the acceleration of a particle moving in a plane is given by Eqn. 1.12. Thus the force $\mathbf{F}$ on the particle m or its equation of motion is

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}_1 = \mathbf{r}_1 m(\ddot{r} - r\dot{\theta}^2) + \boldsymbol{\theta}_1 m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad (4.13)$$

From this it follows that, since the component of force along $\boldsymbol{\theta}_1$, or perpendicular to $\mathbf{r}_1$, is zero, the component of acceleration in this direction must be zero. Hence for any central force

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

This may be written

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

which gives by integration

$$r^2\dot{\theta} = \text{Constant} = k \tag{4.14}$$

Let us consider the significance of the equation $r^2\dot{\theta} = k$. Let the vector $\mathbf{r}$ in Fig. 4.6 rotate through a small angle $\Delta\theta$, as shown in Fig. 4.7. The area ΔA swept out by the vector $\mathbf{r}$ in turning through $\Delta\theta$ would then be

$$\Delta A = \frac{r^2\,\Delta\theta}{2}$$

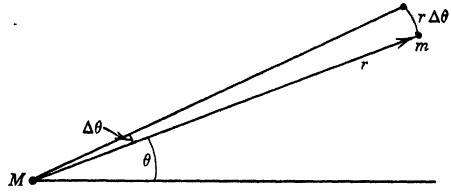


FIG. 4.7 Area swept out by the vector r.

If the angle $\Delta\theta$ is turned through in a time Δt, then the area ΔA is also swept out in this time.

$$\frac{\Delta A}{\Delta t} = \frac{r^2\,\Delta\theta}{2\,\Delta t}$$

and in the limit

$$\frac{dA}{dt} = \frac{r^2\,d\theta}{2\,dt}$$

Thus from Eqn. 4.14

$$r^2\dot{\theta} = 2\frac{dA}{dt} = k \tag{4.15}$$

or the rate at which the small mass m sweeps out an area as it moves about M is a constant. This is Kepler's law of areas in which the small mass represents a planet and the large mass the sun. From what has been said it follows that the law of areas is valid for any type of central force whether of the inverse square type or not.

In order to find the equation of the path in which m moves it is necessary to eliminate the time t and to obtain an equation between r and θ. The equations from which t is to be eliminated are

$$r^2\dot{\theta} = k$$

and from Eqn. 4.13

$$-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\theta}^2) \tag{4.16}$$

A considerable simplification in the elimination of the time from these equations can be introduced by changing the variable to u where

$$r = \frac{1}{u}$$

By differentiation

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

and

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{k}{r^2}$$

$$= -k\frac{du}{d\theta}$$

Similarly

$$\ddot{r} = \frac{d\dot{r}}{dt} = -k\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -k\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -k^2u^2\frac{d^2u}{d\theta^2}$$

Substituting these values in Eqn. 4.16, we have

$$-GMu^2 = -k^2u^2\frac{d^2u}{d\theta^2} - \frac{k^2u^4}{u}$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{k^2} \tag{4.17}$$

A solution of this equation is

$$u = C\cos(\theta + \alpha) + \frac{GM}{k^2}$$

where C and α are the arbitrary constants of integration. Thus the equation of the path is

$$\frac{1}{r} = C\cos(\theta + \alpha) + \frac{GM}{k^2}$$

The constant α may be eliminated by measuring θ from a new axis, making an angle $-\alpha$ with the old axis, Fig. 4.8. In terms of this new axis the equation of the path is

$$\frac{1}{r} = C\cos\theta + \frac{GM}{k^2} \tag{4.18}$$

This is the equation of a conic section: an ellipse, a parabola, or a hyperbola.

Since there is possibly some haziness in your mind about conic sections we shall digress somewhat to discuss them. Suppose that PAP' in Fig. 4.9 is a portion of a conic section of which FAD' is the axis.

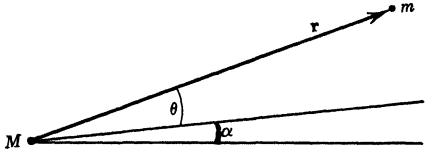


FIG. 4.8 Use of a new axis for the vector r.

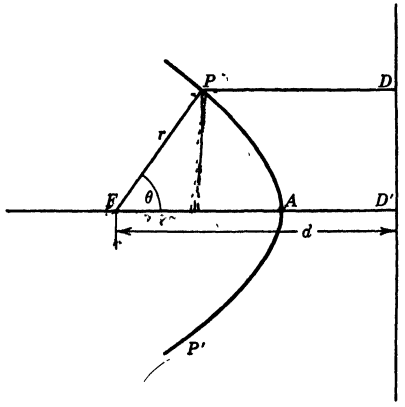


FIG. 4.9 Conic section having focus F, apex A, and directrix DD'.

The conic is symmetrical about the axis. In the diagram F is the focus, A the apex, and DD' the directrix which is perpendicular to the axis. If P is any point on the conic and PD is drawn perpendicular to DD', then the eccentricity e of the conic is defined as

$$e = \frac{FP}{PD}$$

From Fig. 4.9 it may be seen that

$$e = \frac{r}{d - r\cos\theta}$$

or

$$r = ed - er\cos\theta$$

Dividing this equation by edr gives

$$\frac{1}{r} = \frac{\cos\theta}{d} + \frac{1}{ed} \tag{4.19}$$

This is the equation of a conic in polar coordinates. Comparing this equation for a conic with Eqn. 4.18 for the motion of a particle governed by the law of gravitation we have

$$C = \frac{1}{d} \qquad \text{and} \qquad ed = \frac{k^2}{GM}$$

or

$$e = \frac{k^2 C}{GM} \tag{4.20}$$

From analytical geometry we know that a conic section is an ellipse if the eccentricity e is less than one, a parabola if e equals one, and a

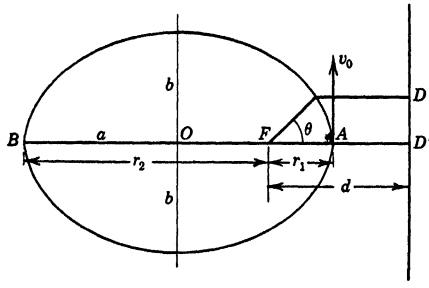


FIG. 4.10 An elliptical orbit of a planet.

hyperbola if e is greater than one. Since the only closed conic section of the above three is the ellipse, it follows that the planets must move in ellipses about the sun with the sun at one of the foci. Comets may move in elliptical, parabolic, or hyperbolic orbits, but it is only when they move in elliptical orbits that they return to be seen again. Thus we see that the law of elliptical orbits follows from Newton's laws.

We shall now proceed to derive Kepler's harmonic law, assuming an elliptical path. Consider the ellipse, Fig. 4.10, as having its focus at F, its semi-major axis $OA = a$, and its semi-minor axis b. If e is the eccentricity of the ellipse, then Eqn. 4.19 applied to the points A and

B at the ends of the major axis gives respectively

$$\frac{1}{r_1} = \frac{1}{d} + \frac{1}{ed}$$

and

$$\frac{1}{r_2} = -\frac{1}{d} + \frac{1}{ed}$$

since, for the point A, $\cos\theta = 1$ and, for the point B, $\cos\theta = -1$ Thus

$$r_1 = \frac{ed}{1+e}$$

and

$$r_2 = \frac{ed}{1-e}$$

Now

$$r_1 + r_2 = 2a = \frac{2ed}{1-e^2} = \frac{2k^2}{GM(1-e^2)}$$

or

$$(1-e^2) = \frac{k^2}{GMa} \tag{4.21}$$

The area of the ellipse may be shown to be πab, and the semi-major and semi-minor axes to be related by $b = a\sqrt{1-e^2}$. If P is the period of revolution of the mass m, then the area πab is swept out in time P. From Eqn. 4.15 the rate of sweeping out of this area is $k/2$ Thus

$$\frac{\pi ab}{P} = \frac{k}{2}$$

Hence

$$P = \frac{2\pi ab}{k} = \frac{2\pi a^2\sqrt{1-e^2}}{k}$$

Using Eqn. 4.21 we have

$$P = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

or

$$P^2 = \frac{4\pi^2 a^3}{GM} \tag{4.22}$$

Thus the square of the period of revolution is proportional to the cube of the semi-major axis or the cube of the mean distance of m from M, which is Kepler's harmonic law. By means of this law the mass of the

sun can be determined if the period and mean distance of one of the planets are known. Similarly, if a planet has a satellite whose period and mean distance are known, then the mass of the planet can be determined.

The harmonic law as given in Eqn. 4.22 is only approximately correct since we have assumed the large mass M to be stationary. The acceleration of mass M or of the sun toward a planet of mass m is Gm/r^2 and of the planet toward the sun is GM/r^2, where r is their distance apart, Fig. 4.11. If the acceleration of the sun is to be made

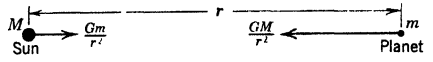


FIG. 4.11 Relative accelerations of the sun M and planet m.

zero, then it must be given an acceleration of Gm/r^2 away from the planet. The acceleration of the planet relative to the sun is then

$$\frac{GM}{r^2} + \frac{Gm}{r^2} = \frac{G}{r^2}(M + m)$$

The corrected form of Kepler's harmonic law is then

$$P^2 = \frac{4\pi^2 a^3}{G(M + m)} \tag{4.23}$$

Thus the law correctly stated should be: The square of the periodic time of a planet about the sun is proportional to the cube of its mean distance from the sun and inversely proportional to the sum of the masses of the planet and sun. Actually this correction is very small for the solar system, since the largest planet, Jupiter, is less than one-thousandth of the mass of the sun, and the earth is about one three-hundred-thousandth of the mass of the sun.

4.9 The Energy of a Planet Moving in its Orbit

We shall again assume the sun with its mass M to be at rest and the planet with its mass m to be moving about the sun. Thus the planet possesses both kinetic and potential energy. If v is the velocity of a planet of mass m, its kinetic energy is $\frac{1}{2}mv^2$ and its potential energy at a distance r from the mass M is $-GMm/r$, from Eqn. 4.12. Since the dissipative forces on the system are negligible, the system is a conservative one and the sum of the kinetic and potential energies is a constant which we shall call E_T. Thus

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = E_T \tag{4.24}$$

The velocity of the planet varies with its distance r from the sun and can be most easily calculated when the planet is at either A or B, Fig. 4.10; that is, when it is at one end of the major axis. For these positions the velocity is perpendicular to the major axis of the conic section. Let v_0 be the velocity when the planet is at A, a distance r_1 from the focus F. From Eqn. 4.14, namely $r^2\dot{\theta} = k$, it follows that the velocity is

$$v_0 = r_1\dot{\theta} = \frac{k}{r_1}$$

From Eqn. 4.18 at $r = r_1$ it follows that

$$\frac{1}{r_1} = \frac{GM}{k^2} + C$$

Hence

$$v_0 = \frac{k}{r_1} = \frac{GM}{k} + Ck$$

The energy equation, Eqn. 4.24, at $r = r_1$ and $v = v_0$ gives

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_1} = E_T$$

or

$$2\frac{E_T}{m} = v_0^2 - 2\frac{GM}{r_1}$$

$$= \left(\frac{GM}{k}\right)^2 + C^2k^2 + 2GMC - 2\frac{G^2M^2}{k^2} - 2GMC$$

$$= -\left(\frac{GM}{k}\right)^2\left(1 - \frac{C^2k^4}{G^2M^2}\right)$$

Using Eqn. 4.20,

$$2\frac{E_T}{m} = -\left(\frac{GM}{k}\right)^2(1 - e^2)$$

Hence

$$E_T = -\frac{m}{2}\left(\frac{GM}{k}\right)^2(1 - e^2)$$

$$e = \sqrt{1 + 2\frac{E_Tk^2}{mM^2G^2}} \tag{4.25}$$

This equation applies to any conic section in which the mass m can move. This equation shows that if the total energy of the planet m in its orbit is negative the eccentricity e is less than one and the planet

moves in an elliptical path. Similarly the orbit is parabolic or hyperbolic according to whether the total energy is zero or positive. In order for the total energy of a planet to be negative its potential energy $-GMm/r$ must be numerically larger than its kinetic energy $mv^2/2$.

PROBLEMS

1. Find the value of the gravitational constant G in the mks system of units, using the value in the cgs system given in the text. Find the force of attraction in poundals between two particles each 1 lb in mass placed 1 ft apart, and also find the force in pounds weight between two particles each 1 slug in mass placed 1 ft apart.

2. Show that the force on a particle inside a closed spherical shell is zero.

3. Show that the potential inside a closed spherical shell is a constant equal to its value at the surface of the shell. If the potential at infinity is taken as zero, find the potential at the surface of a spherical shell of mass m and radius R.

4. Find the mass and mean density of the earth, given that its radius is 6.37×10^8 cm. Use the values given in the text for the acceleration of gravity g and the gravitational constant G

5. Find the vertical velocity a body must have if it is to escape from the earth's gravitational field

6. Find the force in dynes and newtons exerted on a mass of 1 kg placed at a depth of one-quarter of the earth's radius below the surface. Assume the earth to be of constant density.

7. If a hole were bored through a diameter of the earth and a particle dropped into it, then, assuming the earth to be of constant density, find:

a. The velocity of the particle at the center of the earth

b. The time the particle takes to reach the center of the earth.

c. The velocity with which the particle emerges from the other side of the earth.

8. Find the mass of the sun, assuming that the mass of the earth is negligible. The period of the earth about the sun is approximately 365.25 days, and the mean distance of the earth from the sun is 93,000,000 miles.

9. What is the approximate per cent error made in calculating the period of the earth about the sun, if one assumes that the sun is fixed relative to the earth rather than the two rotate about their common center of mass.

10. Assuming the mass of the earth to be 5.95×10^{27} gm, find the mass of the moon if its mean distance from the earth is 239,000 miles or 3.84×10^{10} cm and its period about the earth is 27.3 days.

11. A body falls from a height of 1500 miles toward the earth. If air resistance is neglected but the variation of gravity with height is taken into consideration, find:

a. The velocity with which the body reaches the ground.

b. The time taken to reach the ground.

c. The time taken in falling the last 500 miles to the earth. Assume that the radius of the earth is 4000 miles

12. A bullet is shot vertically upward with a velocity of 3 mi per sec (4.83×10^5 cm per sec). If air resistance is neglected but the variation of gravity with height is considered, find:

a. The height above the earth's surface to which the bullet rises.

b. The time the bullet takes to reach its maximum height.

c. The velocity with which the bullet strikes the ground.

13. The first check Newton made on the law of gravitation was to compare the acceleration of an apple toward the earth with that of the moon toward the earth. He assumed the path of the moon about the earth to be circular and the distance to the moon to be sixty times the mean radius of the earth. Show that the acceleration due to the earth's gravity at the moon is approximately equal to that of the moon in its orbit, each being equal to about 0.27 cm per sec per sec.

14. A particle of mass m moves under a central repulsive force of mb/r^3 and is initially moving at a distance a from the origin of the force with a velocity V at right angles to a. Show that the equation of the path of the particle is

$$r \cos (p\theta) = a$$

where $p^2 = (b/a^2V^2) + 1$.

15. If the mean distance of Mars from the sun is 1.524 times that of the earth from the sun, find the time of revolution in years of the planet Mars about the sun.

16. The period of the planet Jupiter is 11 86 years, and its mean distance from the sun is 483×10^6 miles. A satellite of Jupiter, analogous to our moon, is 261,000 miles from Jupiter, and the satellite has a period of 1 day 18.5 hours From this data show that the ratio of the mass of the sun to the mass of Jupiter is a little larger than a thousand.

17. From the data in problem 16 find the mass of the planet Jupiter.

18. A particle of mass m is attracted by a central force ma/r^3 toward a fixed point. Initially the particle is at a distance c from the fixed point and has a velocity of $\sqrt{a}/c$ in a direction at 45° to c and away from the fixed point. Show that the path of the particle is an equiangular spiral whose equation is $r = ce^{-\theta}$.

19. Two particles having masses of m and M respectively attract each other according to the law of gravitation Initially they are at rest at an infinite distance apart. Show that their relative velocity of approach is

$$\sqrt{\frac{2G(M + m)}{a}}$$

where a is their separation and G is the constant of gravitation.

20. A particle of mass m is attracted by a central force of ma/r^5 toward a fixed point. Initially the particle is at a distance c from the origin and has a velocity of $\sqrt{a/2c^4}$ in a direction at right angles to c. Show that the equation of the orbit is $r = c \cos \theta$.

21. The maximum and minimum velocities of a planet revolving about the sun are 18.3 and 17.7 kilometers per second respectively. Find the eccentricity of the orbit.

22. Assuming that the moon subtends an angle of 0.5° at the earth and is 239,000 miles or 3.84×10^{10} cm away from the earth and has a mass of $\frac{1}{81}$ of the mass of the earth, find the velocity of escape of a particle from the moon.

23. If the eccentricity of a planet's orbit about the sun is 0.02, find:

a. The ratio of the lengths of the major to the minor axes of the orbit of the planet.

b. The ratio of the velocities of the planet when it is at the ends of the major axis of its elliptical orbit.

24. Show that the energy of a planet moving in an elliptical orbit is the same

for all ellipses having the same major axes and does not depend on the shape or eccentricity of the ellipse.

25. Show that, if the force between two masses m and M is that given by the repulsive inverse square law, as for similarly charged particles, then the orbit of m about M is hyperbolic. Assume that M is very much greater than m. (Consider the total energy of the system.)

26. Suppose that a particle of very small mass is at rest at an infinite distance from a large mass M. If the particle m is pulled in toward M by gravitational attraction, show that the orbit of m about M is parabolic.

Show that, if the orbit of m about M is to be an ellipse, then its velocity in its orbit must be less than that acquired in falling from infinity.

27. An alpha particle emitted from radium has a velocity of about 1.5×10^9 cm per sec and is shot out in such a direction as to make a head-on collision with the nucleus of a gold atom. The charge on the alpha particle is 9.6×10^{-10} electrostatic units, and its mass is 6.6×10^{-24} gm. The charge on the nucleus of the gold atom is $79 \times 4.8 \times 10^{-10}$ electrostatic units. The positive charge on the alpha particle and the positive charge on the nucleus of the gold atom repel each other according to the inverse square law of force.

a. Find the closest distance of approach of the alpha particle to the nucleus of the gold atom in the head-on collision.

b. Show that in general the alpha particle would move in a hyperbolic path about the nucleus.

5 · FREE AND FORCED HARMONIC OSCILLATIONS

5.1 Introduction

A great many of the phenomena we observe in everyday life have a periodic motion. Our pulse or our heartbeat, a pendulum clock, many kinds of machinery as well as the seasons and night and day are examples of phenomena repeated at regular intervals of time. In many cases the periodic motion is maintained by a periodic driving force. Without this the oscillations would gradually die out, owing to dissipative forces in the system. For these so-called forced oscillations the amplitude or maximum displacement depends on the frequency of the driving force. There is usually some frequency of the driving force at which the oscillations have their maximum amplitude. The vibrating system and the periodic driving force are then in resonance. This phenomenon of resonance plays an important role in almost every branch of physics. It is encountered in mechanical vibrations, in musical instruments, in the dispersion of light and x-rays, and in alternating-current circuits.

The development of the analysis of periodic phenomena gives one an insight into a portion of the development of physics. It shows how relatively complicated phenomena in light, electricity, and so forth were explained in terms of simple mechanical ideas. Perhaps from what is presented here a partial picture of this unity in the realm of physics may be obtained. As an introduction to the analysis of periodic phenomena we shall first discuss the vibrations of an idealized spring.

5.2 Simple Harmonic Motion of a Mass Hung on an Idealized Spring

By an idealized spring is meant one in which it may be assumed that the mass of the spring is negligible, that there are no dissipative forces tending to decrease the motion of the spring, and that the restoring force exerted by the spring is strictly proportional to its extension. Although these assumptions are never completely justified in practice, nevertheless, for a light spring in which the extensions are kept small and the oscillations are not prolonged over a long time, the errors introduced are not large. At this point one may well ask what is meant by

a light spring, a small extension, and a long time. It should be recognized that these terms are all relative. By a light spring is meant one whose mass is small compared to the mass hung on its lower end; a small extension is one that is small compared to the length of the spring; a long time is one long compared to the time of one oscillation.

Consider the spring, Fig. 5.1, with its upper end rigidly supported and with a mass m or load mg hung on its lower end. The equilibrium position of the spring with the load mg is indicated by the position O. Let O be the origin of the Y coordinates with displacements upward from O being considered positive and below O negative. Suppose that the load is pulled downward a distance $-y$ by an applied force $-F$. There is then an upward restoring force of $+F$ exerted by the spring If the spring obeys Hooke's law, that is, the restoring force is proportional to the displacement, then

$$F = -ky \tag{5.1}$$

where k is a constant called the force constant of the spring. This force constant k can be measured in dynes per centimeter, newtons per meter, or pounds per foot. The negative sign in Eqn. 5.1 indicates that the restoring force exerted by the spring on the mass m is in the opposite direction to the displacement.

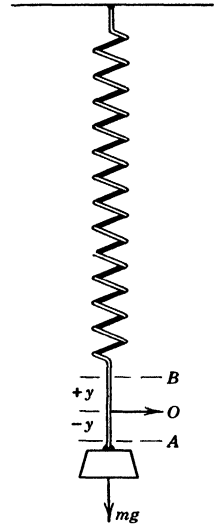


FIG. 5.1 A spring oscillating between the limits A and B.

Suppose that the downward applied force $-F$ is removed. Then the upward restoring force $+F$ accelerates the mass m upward from A to O. Since the restoring force decreases as the mass approaches the equilibrium position at O, the acceleration of the mass decreases until it becomes zero at O. At the equilibrium position O the velocity, momentum, and kinetic energy of the mass m have their maximum values and the mass moves upward to position B where the displacement OB is equal to the original displacement OA. Although at position B the kinetic energy of the mass is zero, its potential energy is mgy. There is a constant interchange of potential energy into kinetic energy and vice versa as the spring oscillates up and down.

By Newton's second law the acceleration $\ddot{y}$ at any instant of time is

$$F = m\ddot{y}$$

Substituting in Eqn. 5.1, it follows that

$$\ddot{y} = -\frac{ky}{m} \tag{5.2}$$

This is the fundamental equation of *undamped simple harmonic motion*. It shows that the acceleration is proportional to and in the opposite direction from the displacement. Table 1 summarizes some of the properties of simple harmonic motion, abbreviated as SHM. The

TABLE 1

Displacement	Restoring Force	Acceler-ation	Velocity	Kinetic Energy	Potential Energy
Positive Maximum $+y_0$	Negative Maximum	Negative Maximum	Zero	Zero	Maximum
Zero	Zero	Zero	Positive or negative Maximum	Maximum	Zero
Negative Maximum $-y_0$	Positive Maximum	Positive Maximum	Zero	Zero	Maximum

maximum displacement OB or OA is called the *amplitude* of the oscillations. The time taken for one complete oscillation, for instance from A to B and back to A, is called the *period* P, and the number of complete oscillations per second is called the *frequency* f of the oscillations From these two definitions it follows that

$$P = \frac{1}{f}$$

In order to obtain the displacement at any instant of time we must solve Eqn. 5.2. This may be done by multiplying both sides of this equation by $2\dot{y}$, giving

$$2\dot{y}\ddot{y} = -\frac{2k}{m} y\dot{y}$$

or

$$\frac{d}{dt}(\dot{y}^2) = -\frac{k}{m}\frac{d}{dt}(y^2)$$

Thus

$$d(\dot{y}^2) = -\frac{k}{m} d(y^2)$$

By integration

$$\dot{y}^2 = -\frac{k}{m}y^2 + C$$

where C is a constant which may be determined if the velocity $\dot{y}$ is known at some displacement y. Hence

$$\frac{dy}{\sqrt{C - (k/m)y^2}} = \pm dt$$

Integration gives

$$\sin^{-1}\sqrt{\frac{k}{mC}}\,y = \pm\sqrt{\frac{k}{m}}\,t + \alpha$$

where α is a constant which may be determined if the displacement y is known at some time t. Thus the displacement at any time is given by

$$y = \sqrt{\frac{mC}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t + \alpha\right)$$

This solution of the second-order differential equation, Eqn. 5.2, contains two arbitrary constants C and α and may be written in the equivalent form

$$y = A\sin\sqrt{\frac{k}{m}}\,t + B\cos\sqrt{\frac{k}{m}}\,t \tag{5.3}$$

Here A and B are the two arbitrary constants

Let us consider a spring that is started from rest at the position $-Y$. At $t = 0$, $y = -Y$ and $\dot{y} = 0$.
Then from Eqn. 5.3

$$-Y = B$$

Differentiating Eqn. 5.3 gives

$$\dot{y} = A\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}\,t - B\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t$$

and, using the initial conditions $\dot{y} = 0$ at $t = 0$, we have

$$0 = A\sqrt{\frac{k}{m}} \text{ or } A = 0$$

Hence the displacement y at any time t with these initial conditions is

$$y = -Y\cos\sqrt{\frac{k}{m}}\,t \tag{5.4}$$

If the period of the motion is denoted by P, then after one period the displacement y has its original value or

$$\cos\left[\sqrt{\frac{k}{m}}\,(t+P)\right] = \cos\left[\sqrt{\frac{k}{m}}\,t + 2\pi\right]$$

so that

$$\sqrt{\frac{k}{m}}\,P = 2\pi$$

or

$$P = 2\pi\sqrt{\frac{m}{k}} \tag{5.5}$$

The displacement under the initial conditions is given by Eqn. 5 4, which may be written

$$y = -Y\cos\frac{2\pi t}{P} \tag{5.6}$$

The quantity $2\pi/P$ is the angular velocity of the motion, omega, ω. The displacement may then be written

$$y = -Y\cos\omega t \tag{5.7}$$

It can be readily verified that the quantity $\sqrt{m/k}$ has the dimensions of time. The force constant k has the dimensions of force divided by distance or $[MLT^{-2}L^{-1}]$ or $[MT^{-2}]$. The dimensions of m/k are $[T^2]$. From Eqn. 5.7 the velocity $\dot{y}$ at any instant is

$$\dot{y} = +Y\omega\sin\omega t \tag{5.8}$$

which may be written

$$\dot{y} = -Y\omega\cos\left(\omega t + \frac{\pi}{2}\right) \tag{5.9}$$

Comparing Eqns. 5.7 and 5.9, we see that the velocity of the mass is 90° ahead of its displacement. When the displacement is zero, the velocity is a maximum. By a second differentiation

$$\ddot{y} = Y\omega^2\cos\omega t = -\omega^2 y$$

Comparing this with the equation of motion, Eqn. 5.2,

$$\ddot{y} = -\frac{k}{m}\,y$$

we see that

$$\omega^2 = \frac{k}{m} \qquad \text{or} \qquad \omega = \sqrt{\frac{k}{m}} \tag{5.10}$$

and the period P is given by

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

which was obtained previously by direct integration.

In general the displacement y at any time t may be written

$$y = y_0 \cos(\omega t + \alpha) \tag{5.11}$$

The angle α is called the *phase angle* inasmuch as it gives the displacement at zero time. At $t = 0$, the displacement y is $y_0 \cos \alpha$ and the velocity $-y_0\omega \sin \alpha$. With the initial conditions used above, namely $y = -Y$ and $\dot{y} = 0$ at $t = 0$, it follows that

$$-Y = y_0 \cos \alpha \qquad \text{and} \qquad -Y\omega \sin \alpha = 0$$

Hence

$$\alpha = 0 \qquad \text{and} \qquad y_0 = -Y$$

Thus the displacement at any time t is

$$y = -Y \cos \omega t$$

which is the result obtained in Eqn. 5.7.

As we have seen, there are several equivalent solutions of the original second-order differential equation of motion. These general solutions must contain two arbitrary constants. For any given set of initial conditions the displacement-time relationship must be unique, since it corresponds to a particular physical situation.

5.3 Kinetic and Potential Energies in Simple Harmonic Motion

The kinetic energy of the oscillating mass at any time is $m\dot{y}^2/2$. Taking the same initial conditions of motion used in deriving Eqn. 5.8, the kinetic energy of the mass m is

$$\tfrac{1}{2}mY^2\omega^2 \sin^2 \omega t = \tfrac{1}{2}kY^2 \sin^2 \omega t$$

since in this problem $\omega^2 = k/m$. The maximum value for the kinetic energy occurs when $\sin^2 \omega t = 1$. With T the symbol for kinetic energy it follows that

$$T_{\max} = \tfrac{1}{2}kY^2 \tag{5.12}$$

The potential energy V at any displacement y is the work done on the mass m in giving it this displacement. From Eqn. 3.6

$$dV = -F\,dy$$

and since the restoring force is

$$F = -ky$$

it follows that

$$V = \int_0^V dV = \int_0^y ky\,dy = \frac{ky^2}{2} \tag{5.13}$$

Thus the potential energy at any time t is

$$V = \frac{k}{2} Y^2 \cos^2 \omega t$$

and has a maximum value of

$$V_{\max} = \frac{k}{2} Y^2$$

Since we have assumed that there are no dissipative forces present, the total mechanical energy of the system should be constant and independent of the time. This is easily proved for

$$\begin{aligned} T + V &= \tfrac{1}{2}kY^2 \sin^2 \omega t + \tfrac{1}{2}kY^2 \cos^2 \omega t \\ &= \tfrac{1}{2}kY^2 \end{aligned}$$

The quantity $\frac{1}{2}kY^2$ is also the maximum value of the kinetic or potential energy, for at the time when the kinetic energy has its maximum value the potential energy is zero, and vice versa.

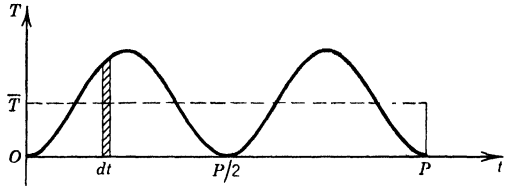


FIG. 5.2 Average value of kinetic energy T over a complete period P.

The average value of the kinetic energy over a complete oscillation will now be considered. In Fig. 5.2 this average value of the kinetic

energy $\bar{T}$ is the ordinate of the rectangle whose area is equal to that enclosed by the sine-squared curve. Mathematically this may be expressed

$$\bar{T} = \frac{1}{P}\int_0^P T\,dt = \frac{1}{P}\int_0^P \frac{k}{2} Y^2 \sin^2 \omega t\,dt$$

Since $2\sin^2 \omega t = 1 - \cos 2\omega t$, this equation may be written

$$\begin{aligned}\bar{T} &= \frac{kY^2}{4P}\int_0^P (1 - \cos 2\omega t)\,dt \\ &= \frac{kY^2}{4P}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^P \\ &= \frac{kY^2}{4}\end{aligned}$$

since $\omega P = 2\pi$.

It can be readily verified in a manner similar to that given above that the average value of $\sin^2 \theta$ or $\cos^2 \theta$ over a complete cycle or angle of 2π radians is $\frac{1}{2}$ and the average value of $\sin \theta$ or $\cos \theta$ over a complete cycle is zero. The time average of the potential energy V over a complete oscillation is similarly given by $kY^2/4$. The sum of these averages is

$$T + V = \frac{kY^2}{2}$$

which is the total energy at any instant of time.

5.4 Effect of the Mass of the Spring

So far we have assumed that the mass of the spring was negligible compared to the oscillating mass hung on its lower end. We shall now take the mass of the spring into account by calculating the total kinetic energy of the system when oscillating. Suppose that the spring is uniformly wound and that its length is l and its mass M. The mass per unit length σ is then given by

$$\sigma = \frac{M}{l}$$

Consider a small portion of the spring of length dy at a height y above the equilibrium position, Fig. 5.3. The mass of this small portion is $\sigma\,dy$. Suppose that the velocity of the lower end of the spring at any instant of time is v. Since the upper end of the spring is always at rest and if we assume that the velocity at any point on

the spring is proportional to its distance from the upper end, it follows that the velocity v_y at any point y is given by

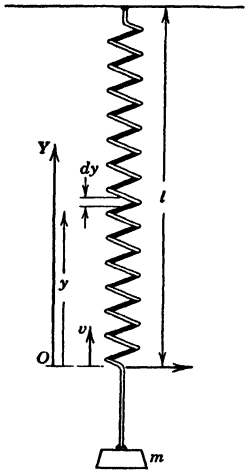


FIG. 5.3 Kinetic energy of spring and mass m.

$$v_y = \frac{v(l - y)}{l}$$

The kinetic energy of the small portion at position y is

$$\frac{1}{2}\sigma {v_y}^2\, dy = \frac{1}{2}\frac{\sigma v^2 (l - y)^2}{l^2}\, dy$$

The total kinetic energy T of the spring and the mass m is

$$\begin{aligned} T &= \frac{1}{2}mv^2 + \frac{1}{2}\frac{\sigma v^2}{l^2}\int_0^l (l - y)^2\, dy \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\frac{\sigma v^2}{l^2}\left[l^2 y + \frac{y^3}{3} - \frac{2ly^2}{2}\right]_0^l \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\sigma v^2 \frac{l}{3} \\ &= \frac{1}{2}v^2\left(m + \frac{M}{3}\right) \end{aligned} \tag{5.14}$$

Thus the fractional part of the mass of the spring partaking in the oscillation is one-third.

5.5 Composition of Simple Harmonic Motions

Let us consider the problem of finding the path of a particle attracted toward a fixed point by a force proportional to its distance from the fixed point. Depending on the initial direction of the velocity of the particle, the path may be a straight line, a circle, or an ellipse. Let O be the center of attraction and A be the point of projection of the particle with a velocity V. Take OA as the direction of the X axis and OB parallel to V as the direction of the Y axis, Fig. 5.4.

At any point P, a distance of r from O, the equation of motion for the particle of mass m is

$$m\frac{d^2\mathbf{r}}{dt^2} = -k\mathbf{r}$$

where k is the force constant.

If PQ is drawn parallel to the Y axis and P has the coordinates x, y, then the force $-k\mathbf{r}$ along PO is equivalent to the vector sum of the

forces $-kx$ and $-ky$ along QO and PQ respectively. Thus

$$m\ddot{x} = -kx$$

$$m\ddot{y} = -ky$$

The solution of these equations, with $\omega = \sqrt{k/m}$, may be written

$$x = X \sin(\omega t + \alpha)$$

$$y = Y \sin(\omega t + \beta)$$

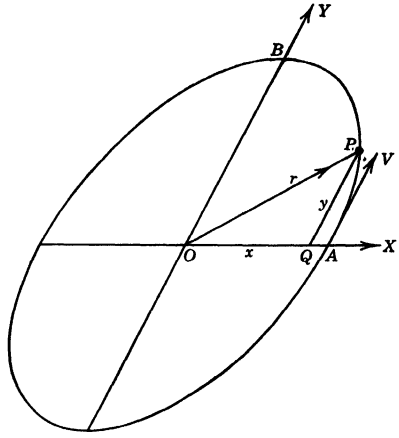


FIG. 5.4 The path of a particle of mass m attracted to a point O by a force proportional to its distance from O.

The initial conditions of motion assumed in Fig. 5.4 are those at $t = 0$: $x = OA = a$, $y = 0$, $\dot{x} = 0$, $\dot{y} = V$. Substituting these values in the above equations gives

$$a = X \sin \alpha$$

and

$$0 = \omega X \cos \alpha$$

Hence

$$\alpha = \frac{\pi}{2} \quad \text{and} \quad a = X$$

and

$$0 = Y \sin \beta \qquad V = \omega Y \cos \beta$$

Hence

$$\beta = 0 \quad \text{and} \quad V = \omega Y \quad \text{or} \quad Y = \frac{V}{\omega}$$

Thus we have

$$x = a \sin\left(\sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right) = a \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$y = \frac{V}{\omega} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

Hence by squaring and adding

$$\frac{x^2}{a^2} + \frac{y^2}{V^2/\omega^2} = 1 \tag{5.15}$$

The path of the moving particle is therefore an ellipse referred to OX and OY as conjugate diameters

The figures produced by two sources of simple harmonic vibrations such as tuning forks or pendulums were first studied by Lissajous about the middle of the last century. A modern simple method of producing these so-called Lissajous' figures is by means of an oscilloscope. In such an instrument the moving particles, the electrons, have a mass of about 9×10^{-28} gm. These small electrically charged particles can be made to oscillate in mutually perpendicular directions by applying alternating voltages to appropriate deflecting plates. By applying voltages of the same frequency and varying the initial conditions it is possible to produce a straight line, a circle, or an ellipse on the viewing screen. If the voltages applied to the X and Y deflecting plates have different frequencies, then very interesting figures may result.

We shall now investigate the case in which the amplitudes of the mutually perpendicular X and Y vibrations are equal and the frequencies are $\omega/2\pi$ and $(\omega + \gamma)/2\pi$ respectively. If the particle is started with maximum displacement along the two axes, then the displacements along the X and Y directions are given by

$$x = A \cos \omega t$$

$$y = A \cos (\omega + \gamma)t$$

Expanding and eliminating ωt gives

$$y = x \cos \gamma t - \sqrt{A^2 - x^2} \sin \gamma t$$

Thus

$$x^2 + y^2 - 2xy \cos \gamma t = A^2 \sin^2 \gamma t \tag{5.16}$$

This may be regarded as the equation of an ellipse whose axes rotate with time. It is easily shown that the ellipse will repeat itself after some time only if the ratio of ω to γ is an integer. Suppose that $(\omega + \gamma)/\omega = 7/6$ or $\gamma = \omega/6$. Then, for one complete vibration along the X direction, there is $7/6$ of a vibration along the Y direction, or, for

each vibration in the X direction, the vibration in the Y direction gains a phase difference of 60°. Thus, when there have been six complete cycles in the X direction, there have been seven complete cycles in the Y direction and the two motions are back at their starting points. This is not possible unless ω/γ is an integer.

To plot the path of the particle it is possible to eliminate t in the above equations, but in practice it is much simpler to take the equations for x and y given above and proceed to solve for x and y by brute force. A partial table of the x and y values for different values of ωt is given below. The values show how $(\omega + \gamma)t$ increases by 60° for each 360° of ωt.

TABLE 2

ωt	$(\omega + \gamma)t$	x	y	Position on Fig 5.5
0°	0°	A	A	a
60°	70°	$0.5A$	$0.34A$	b
360°	360° + 60°	A	$0.5A$	c
2 × 360°	2 × 360° + 120°	A	$-0.5A$	d
3 × 360°	3 × 360° + 180°	A	$-A$	e
4 × 360°	4 × 360° + 240°	A	$-0.5A$	f
5 × 360°	5 × 360° + 300°	A	$0.5A$	g
6 × 360°	6 × 360° + 360°	A	A	h

A graph of this motion is shown in Fig. 5.5 where the path of the particle may be traced.

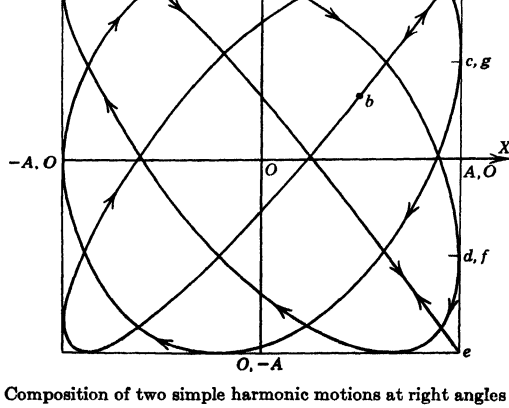


FIG. 5.5 Composition of two simple harmonic motions at right angles and having frequencies in the ratio of 7/6.

5.6 The Simple Pendulum

A simple pendulum consists of a small mass suspended by a light inextensible cord from a fixed support. We shall assume that there are no damping forces so that theoretically once the pendulum is oscillating it continues indefinitely with the same amplitude. Let O in Fig. 5.6 be the fixed point of support, OP the length of the pendulum, and m the mass of the particle at P. The point O is taken as the origin of the radial vector $\mathbf{r}$ which at time t is along OP and makes an angle θ with the vertical through O. We shall now set up the equations of motion of the particle m along and perpendicular to the radius vector OP.

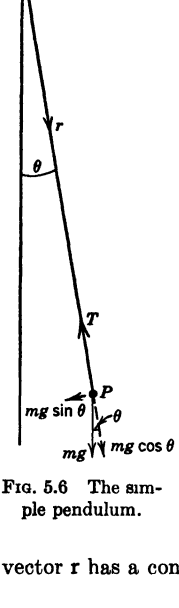


FIG. 5.6 The simple pendulum.

From Eqn. 1.13 the force along the radius vector F_r is

$$F_r = m(\ddot{r} - r\dot{\theta}^2)$$

and this is equal to the components of the applied forces acting on mass m in the direction OP. Thus

$$m(r - r\dot{\theta}^2) = mg \cos \theta - T$$

where T is the tension in the cord at the angle θ Similarly for the transverse force F_θ in the direction of increasing θ we have from Eqn. 1.14

$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -mg \sin \theta$$

Since the string is assumed to be inextensible, the vector $\mathbf{r}$ has a constant magnitude equal to l and it follows that

$$\dot{r} = 0 \qquad \text{and} \qquad \ddot{r} = 0$$

The equations of motion become

$$-ml\dot{\theta}^2 = mg \cos \theta - T \tag{5.17}$$

$$ml\ddot{\theta} = -mg \sin \theta \tag{5.18}$$

In order to make Eqn. 5.18 linear let us assume that θ is so small that sin θ may be set equal to θ in radian measure. This really assumes that θ^3 and higher powers of θ are negligibly small compared to θ since the expansion of sin θ in radian measure is

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

Thus for $\theta = 3°$, sin $3° = 0.05234$ and $3° = 0.05236$ radian. If there-

fore the maximum angular amplitude of the pendulum is a few degrees, we may write Eqn. 5.18 approximately as

$$\ddot{\theta} = -\frac{g}{l}\theta \tag{5.19}$$

This is a second-order differential equation similar to Eqn. 5.2 with the difference that here the variable is an angular displacement rather than a linear one. Following the discussion given earlier we may say that Eqn. 5.19 represents a system oscillating with simple harmonic motion whose period is

$$P = 2\pi\sqrt{\frac{l}{g}} \tag{5.20}$$

The tension T exerted by the cord on the particle depends on the angular displacement as given by Eqn. 5.17, namely

$$T = mg\cos\theta + ml\dot{\theta}^2$$

Thus, when $\theta = 0$, the tension has a maximum value equal to the weight of the particle plus the centripetal force. Since we have assumed no dissipative force, the system is a conservative one and there is conservation of mechanical energy.

We shall now drop the assumption of a small angular amplitude and investigate the period for relatively large amplitudes. To do this we must integrate Eqn. 5.18 or

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Again we employ the trick of multiplying the respective sides of this equation by the identity

$$\dot{\theta}\,dt = \frac{d\theta}{dt}\,dt = d\theta$$

giving

$$\ddot{\theta}\dot{\theta}\,dt = -\frac{g}{l}\sin\theta\,d\theta$$

or

$$\frac{1}{2}d(\dot{\theta}^2) = \frac{g}{l}\,d(\cos\theta)$$

Integrating and using the initial conditions that the pendulum has a maximum angular amplitude of α or $\dot{\theta} = 0$ when $\theta = \alpha$

$$\dot{\theta}^2 = 2\frac{g}{l}(\cos\theta - \cos\alpha) \tag{5.21}$$

Thus

$$\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\, dt$$

The time taken by the pendulum to swing from $+\alpha$ to $-\alpha$ is half of the period of oscillation P_α for the angular amplitude α or

$$\int_{-\alpha}^{+\alpha} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}} \int_0^{P\alpha/2} dt = \sqrt{\frac{2g}{l}} \frac{P_\alpha}{2}$$

Using the transformation

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

the integral becomes

$$\int_{-\alpha}^{+\alpha} \frac{d\theta}{\sqrt{\sin^2\alpha/2 - \sin^2\theta/2}} = \sqrt{\frac{g}{l}}\, P_\alpha \tag{5.22}$$

To evaluate this integral we shall make the substitution

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi \tag{5.23}$$

This is a legitimate substitution inasmuch as the limiting value of θ is α. Hence the limits of $\sin\phi$ are 1 and -1 Let us consider ϕ as the new variable with the limits of $\pi/2$ and $-\pi/2$. Differentiating Eqn. 5.23 we have

$$\frac{1}{2}\cos\frac{\theta}{2}\, d\theta = \sin\frac{\alpha}{2}\cos\phi\, d\phi$$

or

$$d\theta = \frac{2\sin(\alpha/2)\cos\phi\, d\phi}{\sqrt{1 - \sin^2(\alpha/2)\sin^2\phi}}$$

Substituting in Eqn. 5.22 for the period, it follows that

$$\begin{aligned}
\sqrt{\frac{g}{l}}\, P_\alpha &= \int_{-\pi/2}^{+\pi/2} \frac{2\, d\phi}{\sqrt{1 - \sin^2(\alpha/2)\sin^2\phi}} \\
&= \int_{-\pi/2}^{+\pi/2} 2\, d\phi \left(1 - \sin^2\frac{\alpha}{2}\sin^2\phi\right)^{-1/2} \\
&= \int_{-\pi/2}^{+\pi/2} 2\, d\phi \left(1 + \frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \frac{3}{8}\sin^4\frac{\alpha}{2}\sin^4\phi \cdots\right) \\
&= 2\left[\pi + \frac{1}{2}\sin^2\left(\frac{\alpha}{2}\right)\frac{\pi}{2} + \frac{3}{8}\sin^4\left(\frac{\alpha}{2}\right)\frac{3\pi}{8} \cdots\right]
\end{aligned}$$

Hence

$$P_\alpha = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \frac{9}{64}\sin^4\frac{\alpha}{2} \cdot \cdot\right)$$

If the period for an angular amplitude of approximately zero is called P_0 where

$$P_0 = 2\pi\sqrt{\frac{l}{g}}$$

then the period P_α for an angular amplitude α is

$$P_\alpha = P_0\left(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \frac{9}{64}\sin^4\frac{\alpha}{2} \cdot \quad \cdot\right) \tag{5.24}$$

From the three terms in the expansion given in Eqn. 5.24, we obtain for an angular amplitude of 90° a period of

$$P_{90^\circ} = P_0(1 + \tfrac{1}{8} + \tfrac{9}{256})$$
$$= 1.16P_0$$

Even for this amplitude another term in the expansion should be added if real accuracy is desired. However, in most experiments amplitudes as large as 90° are not encountered. For an amplitude of 20° the contribution of the $\sin^2 \alpha/2$ term is 0.0075 and of the $\sin^4 \alpha/2$ term is 0.00013. If the amplitude is smaller than 20°, then the $\sin^4 \alpha/2$ term may be neglected without making an appreciable error. A further simplification can be made by replacing $\sin^2 \alpha/2$ by $\alpha^2/4$. Thus for $\alpha = 20°$ the value of $\sin^2 10°$ is approximately 0.0308 and $\alpha^2/4$ is approximately 0.0305 in radian measure. With these limitations we may write the period

$$P_\alpha = P_0\left(1 + \frac{\alpha^2}{16}\right)$$

and by the binomial theorem to the same degree of accuracy it follows that

$$P_0 = P_\alpha\left(1 - \frac{\alpha^2}{16}\right)$$

If the angular amplitude is 4° or 0.0698 radian, then

$$P_0 = P_{4^\circ}(1 - 0.0003)$$

or

$$P_{4^\circ} = P_0(1 + 0.0003) = 1.0003P_0$$

The periods differ by 3 parts in 10,000. The tension in the cord can now be evaluated for any angular displacement. From Eqn. 5.17 the tension in the cord is

$$T = mg \cos \theta + ml\dot{\theta}^2$$

and, substituting for $\dot{\theta}^2$ from Eqn. 5.21, we have

$$T = mg(3 \cos \theta - 2 \cos \alpha)$$

5.7 Simple Harmonic Motion with Damping

We shall now drop the assumption that there are no dissipative forces acting on the particle which is executing simple harmonic motion. The amplitude of a simple pendulum continuously decreases with time largely owing to air friction, though if the pendulum were placed in a vacuum there would still be some damping or decrease in amplitude due to the lack of perfect rigidity of the supports.

For relatively small velocities of the moving particle it is a fairly accurate assumption that the damping force is proportional to the velocity of the particle. This damping force always opposes the motion or is in the opposite direction to the velocity. Thus, if the velocity of the moving particle is v, the damping force may be set equal to $-Rv$ where R is a constant, namely the damping force per unit velocity.

If at some instant of time the mass m has a displacement y from its equilibrium position and a velocity $\dot{y}$, then by Newton's second law the equation of motion of the mass m in Fig. 5.1 is

$$m\ddot{y} = -ky - R\dot{y}$$

where $-ky$ is the restoring force due to the spring and $-R\dot{y}$ the damping force opposing the motion. Thus

$$m\ddot{y} + R\dot{y} + ky = 0 \tag{5.25}$$

This is a linear homogeneous differential equation of the second order with constant coefficients. It is a linear equation because it contains no powers of y higher than the first; it is homogeneous because it contains no term independent of y or its derivatives; it is of the second order because its highest differential is of the second order. Such equations can be solved relatively simply by assuming an exponential relationship between the variables y and t such as

$$y = Ce^{\lambda t} \tag{5.26}$$

where C and λ are constants depending on the particular problem. By differentiation of Eqn. 5.26

$$\dot{y} = \lambda C e^{\lambda t}$$

and

$$\ddot{y} = \lambda^2 C e^{\lambda t}$$

Substituting these values in the equation of motion, Eqn. 5.25, we have the auxiliary or characteristic equation

$$m\lambda^2 + R\lambda + k = 0$$

Thus the solution of this equation is

$$\lambda = \frac{-R \pm \sqrt{R^2 - 4mk}}{2m} = -\frac{R}{2m} \pm \sqrt{\frac{R^2}{4m^2} - \frac{k}{m}}$$

The displacement of the mass m at any time t is

$$y = e^{-\frac{R}{2m}t}\left[Ae^{+\sqrt{(R^2/4m^2)-(k/m)}t} + Be^{-\sqrt{(R^2/4m^2)-(k/m)}t}\right] \quad (5.27)$$

where A and B are the constants of integration required to satisfy a second-order differential equation.

Equation 5.27 presents three distinct cases according to whether $R^2/4m^2$ is greater than, equal to, or less than k/m. These correspond physically to the following situations.

(a) $$\frac{R^2}{4m^2} > \frac{k}{m}$$

In this situation the damping force is large compared to the restoring force. This results in the displaced mass coming slowly to the equilibrium position but not passing it. The system is said to be *overdamped.*

(b) $$\frac{R^2}{4m^2} = \frac{k}{m}$$

In this the displaced mass comes to its equilibrium position in a minimum time and does not pass beyond the equilibrium position. The system is said to be *critically damped.*

(c) $$\frac{R^2}{4m^2} < \frac{k}{m}$$

Here the damping force is relatively small compared to the restoring force. The motion of the mass m is oscillatory with the amplitude decreasing with each oscillation. This is called *damped oscillatory motion.*

We shall now discuss each of these cases.

5.8 Overdamped Motion: $R^2 > 4km$

For this condition the radical is real and the solution is that given by Eqn. 5.27. The constants of integration A and B are determined from the initial or boundary conditions. Let us assume these to be that at time $t = 0$, $y = y_0$ and $\dot{y} = 0$. That is, the mass m starts from rest with an initial upward displacement of $+y_0$. Substituting these values in Eqn. 5.27 yields

$$y_0 = A + B$$

Differentiating Eqn. 5.27 and for convenience replacing

$$\sqrt{(R^2/4m^2) - (k/m)}$$

by α, we have

$$\dot{y} = -\frac{R}{2m} e^{-(R/2m)t}(Ae^{\alpha t} + Be^{-\alpha t}) + e^{-(R/2m)t}(A\alpha e^{\alpha t} - B\alpha e^{-\alpha t})$$

Substituting $y = 0$ at $t = 0$ gives

$$0 = -\frac{R}{2m}(A + B) + \alpha(A - B)$$

or

$$\frac{R}{2m}\frac{y_0}{\alpha} = A - B$$

Thus the quantities $(A + B)$ and $(A - B)$ can be determined so that A and B can be found. Rather than carry this out with symbols we shall take a numerical example.

Example of overdamped motion. Suppose that a spring has a force constant of 2500 dynes per cm, a damping force constant R of 1000 dyne sec per cm, and a mass of 25 gm hung on the lower end. The problem is to find the displacement y at any time t if the mass is started from rest 5 cm below the equilibrium position. First we must check whether the motion is overdamped or whether R^2 is greater than $4km$. Substituting these numerical values gives $R^2 = 10^6$ and $4km = 2.5 \times 10^5$ so that the motion is overdamped and Eqn. 5.27 is directly applicable. From the data

$$\frac{R}{2m} = 20 \text{ sec}^{-1} \qquad \frac{k}{m} = 100 \text{ sec}^{-2}$$

$$\sqrt{\frac{R^2}{4m^2} - \frac{k}{m}} = \sqrt{300} = 17.3 \text{ sec}^{-1} \quad \text{(approximately)}$$

Substituting these numerical values in Eqn. 5.27 gives

$$y = e^{-20t}(Ae^{17.3t} + Be^{-17.3t})$$
$$= Ae^{-2.7t} + Be^{-37.3t}$$

The second term is negligible for anything but exceedingly small values of t. Nevertheless we cannot neglect it at present. From the initial condition that $y = -5$ cm at $t = 0$

$$-5 = A + B$$

Also

$$\dot{y} = -2.7Ae^{-2.7t} - 37.3Be^{-37.3t}$$

For $\dot{y} = 0$ at $t = 0$

$$0 = -2.7A - 37.3B$$

These equations give

$$A = -5.39 \text{ cm} \qquad B = 0.390 \text{ cm}$$

Thus the displacement at any time t is

$$y = -5.39e^{-2.7t} + 0.390e^{-37.3t}$$

The graph of this equation is shown in Fig. 5.7*a*.

5.9 Critically Damped Motion: $R^2 = 4km$

In this case Eqn. 5.27 for the displacement reduces to

$$y = e^{-(R/2m)t}(A + B)$$

Since the two constants A and B are multiplied by the same time factor they are really equivalent to one constant and as such this equation does not satisfy the second-order differential equation requirements. The solution for this condition found in most texts on differential equations, is

$$y = (A + Bt)e^{-(Rt/2m)} \tag{5.28}$$

That this is a solution may be readily verified by substituting it in the original equation of motion, Eqn. 5.25. The constants of integration, A and B, are obtained from the initial conditions as seen in the following example.

Example of critically damped motion. Let us take the data of the previous example, namely, $k = 2500$ dynes per cm; $m = 25$ gm; and, at $t = 0$, $y = 0$ and $y = -5$ cm. First we must find the value of R_c required for critical damping. Substituting these values in the equation $R_c^2 = 4km$, we have

$$R_c^2 = 25 \times 10^4$$

or

$$R_c = 500 \text{ dyne sec per cm}$$

and

$$\frac{R_c}{2m} = 10 \text{ sec}^{-1}$$

Thus the displacement at any time t is

$$y = (A + Bt)e^{-10t}$$

Substituting the values $y = -5$ cm at $t = 0$ sec gives

$$A = -5$$

Also

$$\dot{y} = Be^{-10t} - 10(A + Bt)e^{-10t}$$

and using $y = 0$ at $t = 0$ gives

$$0 = B - 10A$$

Thus $A = -5$ cm and $B = -50$ cm per sec. The displacement y at any time t for the critically damped motion is

$$y = -5e^{-10t}(1 + 10t)$$

The graph of this equation is given in Fig. 5.7*b*. Comparison of graphs *a* and *b* in Fig. 5.7 shows that when the motion is critically damped the mass m reaches its equilibrium position in a shorter time than when the damping is larger. Although theoretically in both cases the equilibrium position is not reached until after an

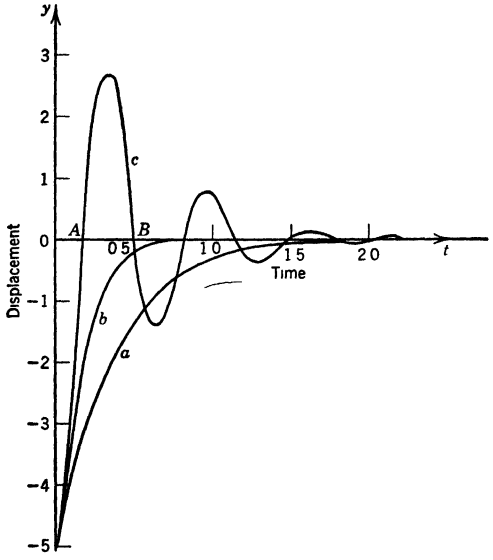


FIG. 5.7 Graph of displacement of mass m on the end of the spring plotted against time. (*a*) Overdamped motion. (*b*) Critically damped motion. (*c*) Damped oscillatory motion.

infinite time, nevertheless for all practical purposes the displacement becomes very close to zero in a finite time. For any damping constant less than 500 dyne sec per cm in our particular example the motion becomes oscillatory. This type of motion we shall now investigate.

5.10 Damped Oscillations: $R^2 < 4km$

In this case the radical $\sqrt{R^2/4m^2 - k/m}$ is the square root of a negative quantity and is therefore imaginary. To overcome this difficulty we shall make use of the $\sqrt{-1}$ represented by the symbol j. Thus Eqn. 5.27 for the displacement becomes

$$y = e^{-Rt/2m}\left[Ae^{j\sqrt{(k/m)-(R^2/4m^2)}\,t} + Be^{-j\sqrt{(k/m)-(R^2/4m^2)}\,t}\right] \quad (5.29)$$

For convenience we shall put

$$\sqrt{\frac{k}{m} - \frac{R^2}{4m^2}} = \omega'$$

The appropriateness of the symbol ω' will appear later when it will be shown to be an angular velocity. Now by Euler's theorem

$$e^{j\omega' t} = \cos \omega' t + j \sin \omega' t$$

$$e^{-j\omega' t} = \cos \omega' t - j \sin \omega' t$$

With these relationships Eqn. 5.29 becomes

$$y = e^{-Rt/2m}[(A + B) \cos \omega' t + j(A - B) \sin \omega' t]$$

For convenience let

$$(A + B) = C \qquad \text{and} \qquad j(A - B) = D$$

Then

$$y = e^{-Rt/2m}\,(C \cos \omega' t + D \sin \omega' t)$$

or

$$y = Ke^{-Rt/2m} \cos (\omega' t - \phi) \quad (5.30)$$

where

$$K \cos \phi = C \qquad \text{and} \qquad K \sin \phi = D$$

The two constants of integration determined from the initial conditions are A and B or C and D or K and ϕ. Which pair is used is immaterial though Eqn. 5.30 with K and ϕ is often the most convenient. From this equation it may be seen that the motion is oscillatory, having a period of $P = 2\pi/\omega'$. Note that the period of oscillation with damping is larger than that without damping. The amplitude of the motion is continuously decreasing owing to the factor $e^{-Rt/2m}$ which is called the damping factor. The principal features of this motion are given in the following example.

Example of damped oscillation. Again we take the data of the previous problem: $k = 2500$ dynes per cm, $m = 25$ gm; but now the damping constant R is equal to 100 dyne sec per cm. The initial conditions are: at $t = 0$, $y = -5$ cm

and $\dot{y} = 0$. To be sure that these data correspond to oscillatory motion we must show that R^2 is less than $4km$. Since $R^2 = 10^4$ and $4km = 2.5 \times 10^5$, the motion is oscillatory and Eqn. 5.30 is applicable. From the data

$$\omega' = \sqrt{\frac{k}{m} - \frac{R^2}{4m^2}} = \sqrt{96} = 9.80 \text{ sec}^{-1} \quad \text{(approximately)}$$

and

$$\frac{R}{2m} = 2 \text{ sec}^{-1}$$

Thus Eqn. 5.30 becomes

$$y = Ke^{-2t} \cos (9.80t - \phi)$$

Substituting the initial conditions $y = -5$ cm at $t = 0$,

$$K \cos \phi = -5$$

Also

$$\dot{y} = -2Ke^{-2t} \cos (9.80t - \phi) - 9.80Ke^{-2t} \sin (9.80t - \phi) \qquad (5.31)$$

At $t = 0$, $y = 0$. Hence

$$0 = -2K \cos \phi + 9.80K \sin \phi$$

Thus

$$\tan \phi = \frac{2}{9.8} = 0.204$$

and

$$\phi = 11° 32' \text{ or } 191° 32'$$

Also

$$K = -\frac{5}{\cos \phi}$$

If $\phi = 11° 32'$, then $K = -5/0.980 = -5.10$ cm, whereas if $\phi = 191° 32'$, then $K = -5/-0.980 = +5.10$ cm. Either value of ϕ can be substituted, but the appropriate sign for K must also be used. Thus we see that the displacement can be written either

$$y = -5.10e^{-2t} \cos (9.80t - 11° 32')$$

or

$$y = 5.10e^{-2t} \cos (9.80t - 191° 32')$$

If the angle ϕ is stated in degrees, then it is probably more convenient for purposes of calculation to express the $9.80t$ in degrees than in radians

$$1 \text{ radian} = 57° 17' 45'' = \text{approximately } 57.3°$$

and

$$9.80t \text{ radians} = 561.5°t$$

The displacement y at any time t is

$$y = -5.10e^{-2t} \cos (561.5°t - 11.54°) \qquad (5.32)$$

A simple check on this equation is to see whether the displacement is -5 cm at $t = 0$. In order to plot the displacement curve shown in Fig. 5.7*c* it is well if one determines the period P of the oscillations:

$$P = \frac{2\pi}{\omega'} = \frac{2\pi}{9.80} = 0.641 \text{ sec}$$

We can then determine at what times the displacement is zero. These times are obtained by setting the cosine function equal to zero. Thus $y = 0$ when $561.5°t - 11.54° = 90°, 270° \cdots (2n + 1)\ 90°$ where n is any integer. From this it follows that at times of approximately 0.181, 0.501, 0.822, 1.14 sec the displacement is zero. Since displacements are zero every half-period, once the first time of zero displacement has been determined, successive times are found by adding 0.3204 sec successively.

It might appear at first sight that maximum values of the displacement would be obtained by setting the cosine function equal to unity. However, this is not correct, for the maximum values of the displacement occur when the velocity y is zero. From Eqn. 5.31 for the velocity $\dot{y}$ at any time t, it follows that $y = 0$ when

$$\tan (9.80t - \phi) = -\frac{2}{9.80} = -0.204$$

The angles having a tangent of -0.204 are $-11° 32', 180° - 11° 32', 360° - 11° 32'$, etc. Hence the times of maximum displacement occur when the term $(561.5°t - 11° 32')$ is equal to one of the above angles, as when

$$561.5t = 0, 180, 360, \text{etc.}$$

or

$$t = 0, 0\ 32, 0.64, 0\ 96, \text{etc.}$$

Notice that these times are not exactly midway between the positions of zero displacement as would be given by setting the cosine function equal to unity. The values of y for these times can now be calculated, and a graph may be filled in free hand with a fair degree of accuracy. A partial table of these values is given below.

TABLE 3

t	$\cos (561.5°t - 11.54°)$	e^{-2t}	y
0 000	0 985	1 000	−5 000
0 181	0 000		0 000
0 321	0 979	0 527	+2.630
0 501	0 000		0.000
0 641	0 978	0 278	−1 385

The graph of this damped oscillatory motion is shown in Fig. 5.7*c*. It is not a true cosine curve partly because of the negative exponential factor $e^{-Rt/2m}$. For this particular example the larger the value of R, up to the critical value of 500 dyne sec per cm, the more warped the curve becomes.

5.11 The Logarithmic Decrement

The amplitude of the oscillations decreases with time owing to the exponential or damping factor $e^{-Rt/2m}$. This decrease in amplitude is often expressed in terms of the logarithmic decrement δ, which is defined as the natural logarithm of the ratio of two maxima a period apart.

$$\delta = \log_e \frac{y_t}{y_{t+P}} = \log_e \left\{ \frac{Ke^{-Rt/2m} \cos (\omega' t - \phi)}{Ke^{-(R/2m)(t+P)} \cos [\omega'(t + P) - \phi]} \right\} \tag{5.33}$$

$$= \log_e e^{RP/2m} = \frac{RP}{2m}$$

For the example given above, the value of the logarithmic decrement δ is

$$\delta = \frac{RP}{2m} = \frac{100}{50} \times 0.641 = 1.282$$

This value may also be obtained by taking the natural logarithm, or 2.303 times the logarithm to the base 10, of two maxima a period apart.

5.12 The Energy Equation with Damping

Since the amplitude decreases with time, it follows that the mechanical energy of the system is being converted into heat energy through the agency of the damping force. The rate of dissipation of energy, or the rate of doing work against the damping force, is

$$F\dot{y} = Ry^2$$

From the law of conservation of energy this must be equal to the rate of loss of mechanical energy. This statement can readily be proved analytically by the equation of motion, Eqn. 5.25. Thus

$$m\ddot{y} + R\dot{y} + ky = 0$$

Multiplying each term by $\dot{y}$ gives

$$my\ddot{y} + R\dot{y}^2 + ky\dot{y} = 0$$

or

$$\frac{m}{2}\frac{d}{dt}(\dot{y}^2) + R\dot{y}^2 + \frac{k}{2}\frac{d}{dt}(y^2) = 0$$

The successive terms in this equation are the time rate of change of kinetic energy, the time rate of doing work by the damping force, and the time rate of change of potential energy. If the kinetic energy is denoted by T, its rate of change by $\dot{T}$, and similarly for the potential energy V, then

$$\dot{T} + \dot{V} = -R\dot{y}^2 \qquad (5.34)$$

or the rate of change of the mechanical energy is equal to the rate of doing work against the damping force. If the damping force is zero, then

$$\dot{T} + \dot{V} = 0$$

or

$$T + V = A, \text{ where } A \text{ is a constant}$$

This equation shows that the law of conservation of mechanical energy applies to an idealized spring oscillating without damping.

Let us calculate the amount of energy lost in the half-cycle from $t = 0.181$ to $t = 0.501$ sec when the mass on the spring moves from its equilibrium position to its first maximum and back to the equilibrium position. Calling the points of zero displacement corresponding to the times given above A and B, we have for the loss of kinetic energy during the half-cycle

$$T_A - T_B = \tfrac{1}{2}m\dot{y}_A^2 - \tfrac{1}{2}m\dot{y}_B^2$$

From Eqn. 5.31 for the velocity $\dot{y}$, it follows that $\dot{y}^2$ is given by

$$\dot{y}^2 = K^2e^{-4t}[4\cos^2(9.80t - \phi) + 9.80^2\sin^2(9.80t - \phi) + 39.2\sin(9.80t - \phi)\cos(9.80t - \phi)]$$

Now at times t_A and t_B the value of the cosine terms are zero and of the sine terms unity. Substituting numerical values:

$$\tfrac{1}{2}my_A^2 - \tfrac{1}{2}m\dot{y}_B^2 = \tfrac{2.5}{2} \times 5.10^2 \times 9.80^2(e^{-0.724} - e^{-2.004})$$

$$= 10{,}929 \text{ ergs}$$

This loss of energy is equal to the work done against the damping force. From Eqn. 5.34 we have

$$d(T + V) = -R\dot{y}^2\,dt$$

At the points A and B the potential energy is zero since the displacement is zero. Hence by integration

$$T_A - T_B = -\int_B^A R\dot{y}^2\,dt$$

Now this integral which represents the work done against the damping force can be evaluated from mathematical tables, and it will be found to be equal to the loss in kinetic energy in the interval.

It is interesting to note that the maximum kinetic energy does not occur at the times corresponding to zero displacement but slightly before these times. You will be asked to evaluate these times in a problem at the end of this chapter.

5.13 Forced Oscillations of a Damped Oscillator

In every oscillating system there is dissipation of mechanical energy, and consequently, if the oscillations are to be maintained indefinitely, energy must be supplied to the system. The slowly falling weight in a grandfather's clock or the spring in a watch or clock are means of supplying energy to the system to maintain the oscillations. In these the energy is supplied by some mechanism which produces a periodic driving force.

Suppose that a spring is subjected to a periodic force $F_0 \cos \omega t$, where F_0 is the maximum value of the applied force and $f = \omega/2\pi$ is its frequency. Then for a mass m hung on the lower end of a spring whose force constant is k and whose damping constant is R the equation of motion is

$$m\ddot{y} + R\dot{y} + ky = F_0 \cos \omega t \qquad (5.35)$$

Once a steady state is set up, the system will oscillate with the same frequency as the applied force. This is a necessary physical condition of the oscillations. Let us therefore try as a solution of Eqn. 5.35

$$y = A \cos(\omega t - \theta) \tag{5.36}$$

where θ is the angle by which the displacement y lags behind the impressed force $F_0 \cos \omega t$ and A is the amplitude of the oscillations. By differentiation of Eqn. 5.36 we have

$$\dot{y} = -A\omega \sin(\omega t - \theta)$$

$$\ddot{y} = -A\omega^2 \cos(\omega t - \theta)$$

Substituting these values in Eqn. 5.35 and collecting terms gives

$$\cos \omega t(-m\omega^2 A \cos\theta + R\omega A \sin\theta + kA \cos\theta - F_0) + \sin \omega t(-m\omega^2 A \sin\theta - R\omega A \cos\theta + kA \sin\theta) = 0$$

If this equation is to be satisfied for all values of the time t, then the coefficients of the $\cos \omega t$ and of the $\sin \omega t$ terms must each be zero Hence

$$A \cos\theta(k - m\omega^2) + R\omega A \sin\theta - F_0 = 0 \tag{5.37}$$

and

$$-R\omega A \cos\theta + A \sin\theta(k - m\omega^2) = 0 \tag{5.38}$$

From Eqn. 5.38 the phase angle θ between the displacement and the applied force is given by

$$\tan\theta = \frac{\omega R}{k - m\omega^2} \tag{5.39}$$

From Eqn. 5.39 it follows that

$$\sin\theta = \frac{\omega R}{\sqrt{(\omega R)^2 + (k - m\omega^2)^2}}$$

and

$$\cos\theta = \frac{k - m\omega^2}{\sqrt{(\omega R)^2 + (k - m\omega^2)^2}}$$

Substituting these values in Eqn. 5.37 gives for the amplitude A of the oscillations

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\omega R)^2}} \tag{5.40}$$

Hence the solution given by Eqn. 5.36 may be written

$$y = \frac{F_0 \cos(\omega t - \theta)}{\sqrt{(k - m\omega^2)^2 + (\omega R)^2}} \tag{5.41}$$

That this is a solution of the equation of motion, Eqn. 5.35, may be verified by substitution. However, it cannot be a complete solution since it contains no arbitrary constants. The equation of motion, Eqn. 5.35, is a linear, second-order, inhomogeneous differential equation with constant coefficients. The inhomogeneity arises from the term $F_0 \cos \omega t$, which does not contain the dependent variable y or any of its derivatives. The related homogeneous equation is Eqn. 5.25, which does not contain the applied force term. In the theory of differential equations the related homogeneous equation is called the auxiliary equation and its solution the complementary function. The complementary function in this problem is Eqn. 5.27. A solution of the inhomogeneous equation is called the particular integral and is given above in Eqn. 5.41. The complete solution of an inhomogeneous equation is the sum of the complementary function and the particular integral. The proof of this follows from the linearity of the differential equation.

If y_1 is the complementary function or a solution of

$$my + R\dot{y} + ky = 0$$

and y_2 is the particular integral or a solution of Eqn. 5.35, then the complete solution of Eqn. 5.35 is the sum of these or

$$y = y_1 + y_2$$

For this to be a solution

$$m\ddot{y}_1 + R\dot{y}_1 + ky_1 + m\ddot{y}_2 + Ry_2 + ky_2 - F_0 \cos \omega t = 0$$

and this follows from the fact that

$$m\ddot{y}_1 + R\dot{y}_1 + ky_1 = 0$$

and

$$m\ddot{y}_2 + R\dot{y}_2 + ky_2 - F_0 \cos \omega t = 0$$

For this problem, in which the independent variable is the time t, the complementary function is called the *transient solution* inasmuch as its value decreases with time owing to the damping factor. The particular integral is called the *steady-state solution*. Thus the complete solution of our problem is the sum of the transient solution y_t and the steady-state solution y_{ss}. The initial or boundary conditions must be applied to the complete solution:

$$y = y_t + y_{ss}$$

The choice of the solution for the transient depends on the particular value of the damping force constant R. An example will probably make this point clearer.

Example of a forced oscillation. Let us use the data for the spring of the previous problem, namely $k = 2500$ dynes per cm, $m = 25$ gm, and a damping constant R of 200 dyne sec per cm together with an applied force F_0 of 12,500 dynes having a frequency corresponding to $\omega = 8$ radians per sec. The problem is to find the displacement y at any time t if, at time $t = 0$, $y = 0$ and $y = -5$ cm.

First we must examine the transient solution and determine whether this motion is overdamped, critically damped, or oscillatory This depends on the relative sizes of k/m and $R^2/4m^2$. From the data we find that

$$\frac{k}{m} = 100 \text{ sec}^{-2} \qquad \frac{R^2}{4m^2} = 16 \text{ sec}^{-2}$$

Since k/m is greater than $R^2/4m^2$, the motion is oscillatory and the solution is given by Eqn. 5.30. Thus

$$y_t = Ke^{-Rt/2m} \cos (\omega' t - \phi)$$

where

$$\omega' = \sqrt{\frac{k}{m} - \frac{R^2}{4m^2}} = \sqrt{84} = 9.17 \text{ sec}^{-1} \quad \text{(approximately)}$$

The steady-state solution is given by Eqn. 5.41, and the phase angle θ is given by

$$\tan \theta = \frac{\omega R}{k - m\omega^2} = 1.778$$

and

$$\theta = 60° 39' = 1.058 \text{ radians}$$

The complete solution using the numerical values is

$$y = Ke^{-4t} \cos (9.17t - \phi) + 6.80 \cos (8t - 1.058)$$

The initial condition $y = -5$ cm at $t = 0$ gives

$$-5 = K \cos \phi + 6.80 \cos (1.058)$$

or

$$K \cos \phi = -5 - 3.33 = -8.33 \text{ cm}$$

Also

$$\dot{y} = -4Ke^{-4t} \cos (9.17t - \phi) - 9.17Ke^{-4t} \sin (9.17t - \phi) - 8 \times 6.80 \sin (8t - 1.058)$$

At $t = 0$, $\dot{y} = 0$. Hence

$$0 = -4K \cos \phi + 9.17K \sin \phi + (54.4 \times 0.872)$$

or

$$K \sin \phi = \frac{(-47.44 - 33.32)}{9.17} = -8.81 \text{ cm}$$

Hence

$$\tan \phi = 1.058$$

and

$$\phi = 0.810 \text{ radian} = 46° 36'$$

or

$$\phi = 3.952 \text{ radians} = 226° 36'$$

and

$$K = \pm\sqrt{(8.81)^2 + (8.33)^2} = \pm 12.12$$

Now $K \sin \phi$ is a negative quantity. Hence, if we choose the angle ϕ as 46° 24′, then we must take the negative value of K. This is consistent with $K \cos \phi$ being negative. Hence the displacement of the mass m from its equilibrium position at any time t is

$$y = -12.12e^{-4t} \cos (9.17t - 0.810) + 6.80 \cos (8t - 1.058)$$

where all the angles are expressed in radians We see that the first term of this equation representing the transient decreases rapidly with time, for after 2.5 sec the factor e^{-4t} is equal to 0.00005 of its value at zero time. After a few seconds

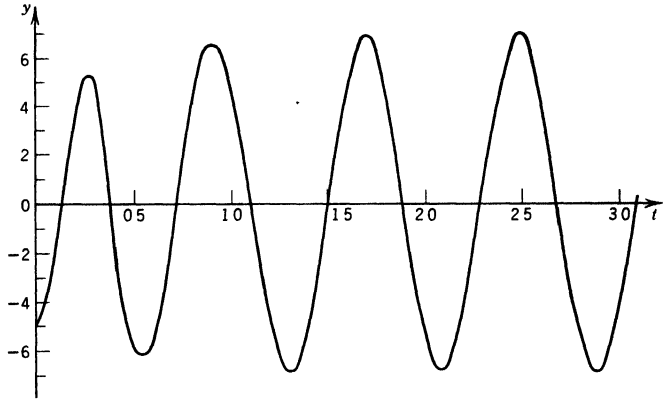


FIG. 5.8 The displacement y of a spring undergoing forced oscillations plotted against the time t.

the contribution from the transient term is effectively zero, and then only the steady-state term remains, which has the same frequency as the applied force. In the steady state the displacement y lags behind the applied force by 1.058 radians or 60° 39′. A plot of this motion is shown in Fig. 5.8. From this it can be seen that the effect of the transient is negligible after about 1 sec.

5.14 Phase Difference between Displacement and Applied Force

As we have seen in Eqn. 5.36, the displacement y of the mass m in the steady state of motion lags behind the applied force by an angle θ which is given in Eqn. 5.39 as

$$\tan \theta = \frac{\omega R}{k - m\omega^2} \tag{5.39}$$

The quantity $\sqrt{k/m}$ is the angular velocity ω_0 of the system vibrating without damping. This we shall call the *natural angular velocity* of the undamped simple harmonic motion of the system, and the corresponding frequency f_0 is

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{5.42}$$

With this notation the phase angle θ may be written

$$\tan\theta = \frac{\omega R}{m(\omega_0^2 - \omega^2)} = \frac{fR}{2\pi m(f_0^2 - f^2)} \tag{5.43}$$

where f is the frequency of the applied force.

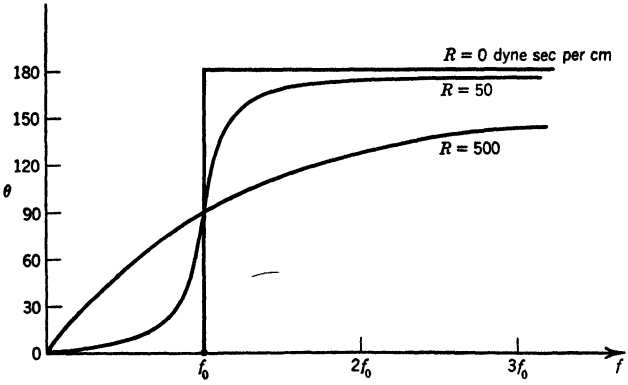


FIG. 5.9 Lag of the displacement θ behind the applied force plotted against the frequency f for different values of the resistance force constant.

As the applied frequency f is increased from zero to the natural frequency f_0 of the system, the phase angle θ increases from zero to 90°. When f is larger than f_0, then $\tan\theta$ is negative and θ increases beyond 90° and approaches 180° as f becomes very large compared to f_0. Thus, when the applied frequency is very large, the displacement lags behind the impressed force by 180°. In other words the displacement and the force are in opposition.

The change in phase with frequency is shown in Fig. 5.9 using the data for the spring problem given earlier, namely $m = 25$ gm, $k = 2500$ dynes per cm. Three values of the damping constant R are used: 0,

50, and 500 dyne sec per cm. The natural frequency f_0 of the system is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{10}{2\pi} = 1.59 \text{ cycles per sec}$$

When R is equal to zero, the angle θ changes abruptly from 0 to 180° where the applied frequency f becomes equal to the natural frequency f_0 of the system. The proof of this is given as a problem at the end of the chapter.

The lag of the displacement relative to the applied force can be simply demonstrated by swinging a meter stick, as a pendulum, about an axis near one end. If the axis is oscillated horizontally, then we notice that the displacement of the lower end is in an opposite direction to the applied force when the frequency f of the applied force is greater than the natural frequency f_0 of the stick. Initially the displacement is in the same direction as the applied force, and it is only later when the steady state is set up that the displacement is in the opposite direction to the applied force.

5.15 Displacement Resonance in Forced Oscillations

The amplitude of the steady-state oscillations varies with the frequency of the impressed force. There is some frequency called the resonance frequency at which the amplitude becomes a maximum. This resonance frequency can be recognized in many vibrating systems. Automobiles, especially the older types, frequently have some part which vibrates violently at a particular speed. Below or above this speed the vibrations are small and usually not noticeable. In some cases such as the marching in step of soldiers crossing a long bridge the increase in amplitude of the vibrating system may produce disastrous results, whereas in a child's swing or the tuning of a radio a large amplitude may be what is desired. If two tuning forks of the same frequency are placed close together and one is set in vibration, the other will commence to vibrate. There are important examples of resonance in sound and in light. In fact resonance can occur in any system capable of vibrating unless the damping force is too large.

The amplitude A of the oscillations at any applied frequency $f = \omega/2\pi$ is given by Eqn. 5.40 as

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\omega R)^2}}$$

The angular velocity, $_r\omega_y$, at which the displacement has its maximum amplitude, or at which there is displacement resonance, can be found

either by plotting A against ω and determining the value of ω at which the maximum amplitude occurs or analytically by determining the value of ω at which $\partial A/\partial\omega = 0$. This we find gives $\partial A/\partial\omega = 0$ when

$$R^2 - 2km + 2m^2{}_r\omega_y{}^2 = 0$$

Thus

$$_r\omega_y = 2\pi_r f_y = \sqrt{\frac{k}{m} - \frac{R^2}{2m^2}} \tag{5.44}$$

From this equation we see that there can be no resonance if $R^2/2m^2$ is greater than k/m; that is, if the damping of a system is large, then the amplitude A continually decreases as ω increases.

Figure 5.10 represents the variation of amplitude with ω for the data given in the earlier problem, namely $m = 25$ gm, $k = 2500$ dynes per cm, and $F_0 = 12{,}500$ dynes. The smaller the value of R, the sharper

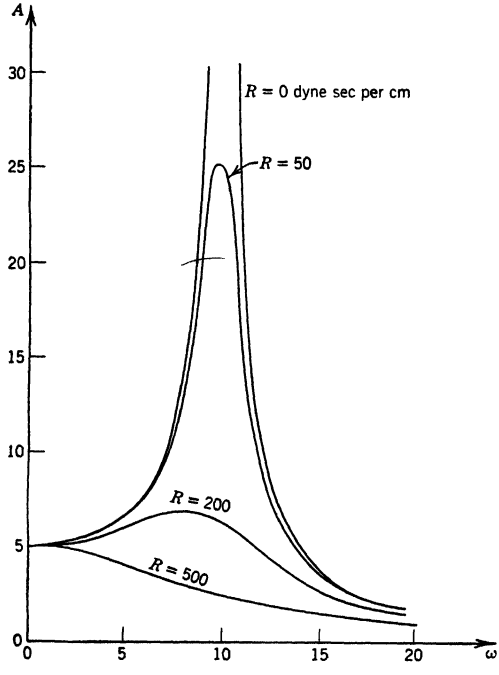


FIG. 5.10 Variation of the amplitude A plotted against the angular velocity ω of the forced oscillations—displacement resonance.

the resonance or the tuning of the circuit to the applied force. Equation 5.44 shows that the resonance frequency varies with the value of the damping force constant R. It can be seen from Eqn. 5.40 that the amplitude becomes infinite when the applied frequency becomes equal to the resonance frequency and the system has zero damping. When the damping constant R is not zero and $\omega = {}_r\omega_y$, then there is a maximum amplitude of ${}_rA_y$. The amplitude at which resonance takes place is

$$ {}_rA_y = \frac{F_0}{R\sqrt{\left(\frac{k}{m} - \frac{R^2}{4m^2}\right)}} $$

5.16 Velocity Resonance in Forced Vibrations

The velocity of the moving particle attached to the spring executing forced vibrations varies from zero to a maximum in a manner somewhat similar to the displacement. However, the frequency of the applied force that produces the maximum velocity is not the same as that which produces the maximum displacement. We shall now determine the frequency necessary to produce the maximum value of the velocity, or to produce velocity resonance.

The velocity $\dot{y}$ of the moving particle in the steady state at any instant of time is given by differentiating Eqn. 5.41. Thus

$$ \dot{y} = \frac{-F_0\omega \sin(\omega t - \theta)}{\sqrt{(k - m\omega^2)^2 + (\omega R)^2}} \tag{5.45} $$

The amplitude of the velocity $\dot{A}$ is the maximum value of $\dot{y}$ obtained by setting the sine term equal to unity.

$$ \dot{A} = \frac{F_0\omega}{\sqrt{(k - m\omega^2)^2 + (\omega R)^2}} \tag{5.46} $$

In order to obtain the frequency at which the velocity amplitude $\dot{A}$ is a maximum one finds $(\partial\dot{A}/\partial\omega)$ and sets the result equal to zero. Carrying this out we find that $\partial\dot{A}/\partial\omega = 0$ when $(k - m\omega^2) = 0$. Thus the angular velocity, ${}_r\omega_v$, required for velocity resonance is given by

$$ {}_r\omega_v = \sqrt{\frac{k}{m}} \tag{5.47} $$

This is the value of the angular velocity of the natural oscillations of the free system without damping as given in Eqn. 5.10. The maxi-

mum velocity $\dot{A}$ is obtained from Eqn. 5.46:

$$\dot{A}_{max} = \frac{F_0}{R} \tag{5.48}$$

It is interesting to note that the angular velocity at which velocity resonance occurs does not depend on the damping force whereas it

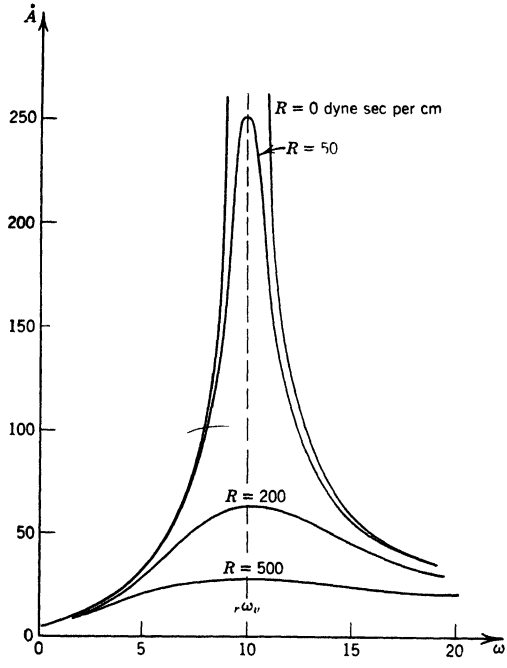


FIG. 5.11 Variation of the velocity amplitude $\dot{A}$ plotted against the angular velocity ω of the forced oscillations—velocity resonance.

does for displacement resonance. Of course the value of the velocity at velocity resonance does depend on the damping force as given by Eqn. 5.48. Figure 5.11 shows the variation of the maximum velocity with the frequency of the applied force based on the data of the previous problem.

5.17 Phase Relations between Displacement and Velocity

In the steady state of the applied force oscillations the displacement and velocity differ in phase by 90° as may be seen from the appropriate equations. For the steady state the displacement is given by

$$y = A \cos (\omega t - \theta)$$

and the velocity by

$$\dot{y} = -A\omega \sin (\omega t - \theta)$$

$$= A\omega \cos \left(\omega t - \theta + \frac{\pi}{2}\right)$$

Thus at any time the velocity leads the displacement by 90° as it does for simple harmonic motion.

For convenience let $\theta - (\pi/2) = \gamma$. Then

$$\tan \theta = \frac{-1}{\tan \gamma}$$

and

$$\dot{y} = A\omega \cos (\omega t - \gamma)$$

Since by Eqn. 5.39 we have

$$\tan \theta = \frac{\omega R}{k - m\omega^2}$$

then

$$\tan \gamma = \frac{m\omega^2 - k}{\omega R} \tag{5.49}$$

Since the applied force at any instant of time t is given by $F = F_0 \cos \omega t$, it follows that the velocity lags behind the applied force by the angle γ.

The angular velocity at velocity resonance is

$$_r\omega_v{}^2 = \frac{k}{m}$$

so that when the applied force has this angular velocity

$$\tan \gamma = 0 \qquad \text{and} \qquad \gamma = 0$$

That is, at velocity resonance the velocity and applied force are in phase whereas the displacement lags behind the applied force by 90°. Velocity resonance plays an important role in the theory of alternating-current circuits since the current in such a circuit is analogous to the velocity in a mechanical system.

5.18 Energy Relationships in Forced Oscillations

In the steady state energy is being supplied to the oscillatory system by the applied force $F_0 \cos \omega t$ and is being dissipated through the agency of the damping force. From the general law of conservation of energy it must follow that in the steady state of motion the rate at which energy is being supplied to the system is equal to the rate at which it is being dissipated. We shall now prove this analytically.

The instantaneous value of the time rate of doing work $\dot{W}$ or the rate at which energy is being supplied to the system is the product of the force and velocity at the same instant of time or

$$\begin{aligned} \dot{W} &= \dot{y} F_0 \cos \omega t \\ &= A\omega F_0 \cos (\omega t - \gamma) \cos \omega t \\ &= A\omega F_0 (\cos^2 \omega t \cos \gamma + \tfrac{1}{2} \sin 2\omega t \sin \gamma) \end{aligned} \tag{5.50}$$

The average value of $\dot{W}$ over a complete cycle or several complete cycles may easily be obtained from the above expression. As was proved earlier in section 5.3 the average value of $\cos^2 \omega t$ over a complete cycle or period is $\frac{1}{2}$ and the average value of $\sin 2\omega t$ is zero. Hence the average rate at which energy is supplied to the system is

$$\bar{\dot{W}} = \frac{A\omega F_0 \cos \gamma}{2} \tag{5.51}$$

This expression may be rewritten in different terms by substituting the value for A from Eqn. 5.40 and noting that Eqn. 5.49 gives for $\cos \gamma$ the value

$$\cos \gamma = \frac{\omega R}{\sqrt{(m\omega^2 - k)^2 + (\omega R)^2}}$$

Hence

$$\bar{\dot{W}} = \frac{{F_0}^2 \cos^2 \gamma}{2R} \tag{5.52}$$

where γ is the angle by which the velocity lags the applied force. The variation of $\bar{\dot{W}}$ with the angular velocity or frequency of the applied force is shown in Fig. 5.12 using the data of the earlier problem. Since at velocity resonance the angle γ is zero, it follows that at this resonance frequency the power supplied to the system is a maximum. This is to be expected, since at this frequency the velocity and kinetic energy have their maximum values.

The instantaneous rate of dissipation of energy D is the instantaneous rate of doing work against the damping force. The rate of

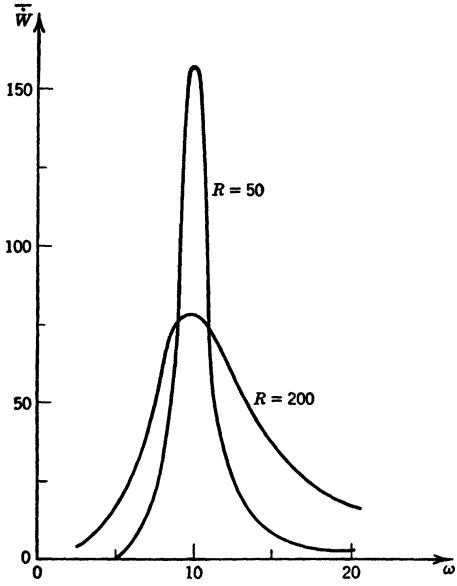


FIG. 5.12 Average rate of dissipation of energy $\overline{W}$ plotted against the angular velocity ω of the forced oscillations.

doing work against the damping force is the product of the damping force $R\dot{y}$ and the velocity $\dot{y}$ at the same instant. Thus

$$\begin{aligned} D &= R\dot{y}^2 \\ &= RA^2\omega^2 \cos^2(\omega t - \gamma) \end{aligned} \tag{5.53}$$

The average rate of dissipation of energy over a complete cycle is

$$\bar{D} = \frac{RA^2\omega^2}{2} \tag{5.54}$$

which by substituting for the amplitude A gives

$$\bar{D} = \frac{F_0^2 \cos^2 \gamma}{2R}$$

As predicted from the law of conservation of energy this is equal to the power supplied to the system by the applied force.

5.19 Forced Vibrations in Other Fields of Physics

Since almost every branch of physics deals with motion of some kind, it is not surprising that some of the ideas developed for mechanics are of value in other branches of physics. It is neither desirable nor possible to present a long list of such examples in this text nor to give the complete theory in any one example. Frequently some of these systems are considered as analogous to a mechanical system. However, though analogies are very useful one has to be careful. Often analogies are drawn on the basis of the similarity of equations and not on fundamental concepts. For instance it is common in alternating-current theory to consider the inductance L as analogous to the mass m in the mechanical case, the capacitance C to the elastance $1/k$ or the inverse of the force constant of the spring, the current i to the velocity $\dot{y}$, the charge q to the displacement y, the resistance R to the damping force constant R, or the electromotive force E to the force F. These analogies are based on the similarity of the equations of motion, for the fundamental concepts in electricity are not the same as those in mechanics nor are they the only possible set of analogies.

The following examples may be omitted without any loss to the subject of mechanics.

a. An alternating-current series circuit. Let us consider a circuit composed of a resistance R, an inductance L, and a capacitance C placed in series with an alternating electromotive force of $E_0 \cos \omega t$, Fig. 5.13. An alternating current is set

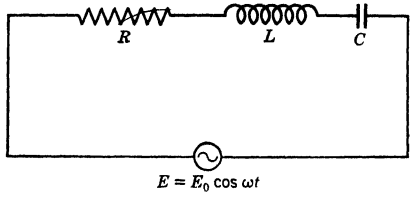


FIG. 5.13 An alternating current circuit consisting of a resistance R, an inductance L, and a capacitance C placed in series

up in the circuit, and, if at any instant of time the current is i, then by Kirchoff's laws the equation for this circuit is given by

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E_0 \cos \omega t \tag{5.55}$$

Since the current i may be considered as the time rate of change of the charge at any point in the circuit, then

$$i = \frac{dq}{dt}$$

and Eqn. 5.55 may be written in terms of the charge q as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \cos \omega t \tag{5.56}$$

This is an inhomogeneous linear differential equation of the second order similar to Eqn. 5 35 for the forced oscillations of a spring. The analogous quantities mentioned above are immediately obtained by comparing Eqn. 5.35 with Eqn. 5.56.

Since it is the current i with which one is usually concerned in an electrical circuit, then we must compare this with the velocity $\dot{y}$ of the oscillating mass. The resonance angular velocity ${}_r\omega_v$ given by Eqn. 5.47 becomes for this electrical circuit

$${}_r\omega_v = \sqrt{\frac{1}{LC}}$$

or

$${}_rf_v = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \tag{5.57}$$

where ${}_rf_v$ is the resonance frequency at which the current reaches its maximum value. The value of this maximum current I_0 given by the analogous Eqn. 5.48 is

$$I_0 = \frac{E_0}{R} \tag{5.58}$$

In alternating-current theory the so-called root-mean-square values of current and voltage are used If these are denoted by I and E respectively, then by the definition of the root-mean-square values

$$I = \frac{I_0}{\sqrt{2}} = 0.707 I_0$$

and

$$E = 0\,707 E_0$$

so that Eqn. 5.58 may be written

$$I = \frac{E}{R}$$

When the frequency of the applied voltage is the resonance frequency given above, then the phase angle γ, Eqn. 5.49, is zero, that is, the applied voltage and current are in phase at every instant of time

When the applied voltage E has an angular velocity ω, then from Eqn. 5.46 the current I is given by

$$I = \frac{E\omega}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + (\omega R)^2}} \tag{5.59}$$

In the electrical case one is interested in the impedance Z of the circuit, which is the ratio of the current to the voltage, or

$$Z = \frac{E}{I} = \frac{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + (\omega R)^2}}{\omega} = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} \tag{5.60}$$

The angle γ by which the current lags behind the applied voltage is given by Eqn. 5.49:

$$\tan \gamma = \frac{m\omega^2 - k}{\omega R} = \frac{L\omega^2 - (1/C)}{\omega R} = \frac{L\omega - (1/\omega C)}{R} \tag{5.61}$$

If one cares to follow this analogy further, one can show relatively easily that the voltage and current are in phase in the resistance, that the voltage leads the current by 90° in the inductance and lags by 90° behind the current in the capacitance. These results are usually represented as in Fig. 5.14. In this diagram the voltage

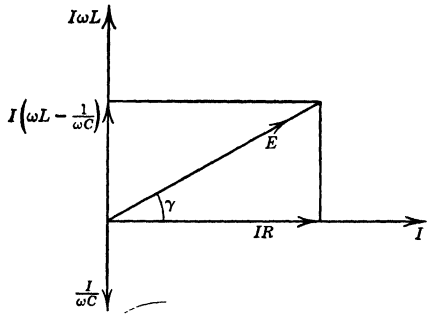


FIG. 5.14 A diagram showing the magnitude and direction of the voltages in an alternating-current circuit.

drop in the resistance is IR in phase with the current, the voltage drop in the inductance is $IL\omega$, leading the current by 90°, and in the capacitance is $I/\omega C$, lagging the current by 90°.

We could have analyzed the mechanical oscillation system by making use of imaginaries. Then the applied force $F_0 \cos \omega t$ would be the real part of the quantity $F_0 e^{j\omega t}$, where j represents $\sqrt{-1}$. By Euler's theorem

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

where $\cos \omega t$ is the real part and $j \sin \omega t$ is the imaginary part of the complex quantity $e^{j\omega t}$. A complex quantity such as $2 + 3j$ can be represented graphically by employing an axis of reals and an axis of imaginaries as shown in Fig. 5.15. In this mode of representation, multiplying a real number by j corresponds to rotating the point through 90°, multiplying by j^2 or -1 to rotating the point through 180°, multiplying by j^3 to rotating the point through 270°, and by j^4 or $+1$ to rotating the point through 360° or 0°. Thus we may represent both the magnitude and the direction of a complex quantity in the complex plane.

For example, the voltage drop in the resistance R is IR in phase with the current,

the voltage drop in the inductance is written $(j\omega L)I$ and leads the current by 90°, and the voltage drop in the capacitance is written as $-jI/\omega C$ or $I/j\omega C$. The applied voltage E may then be expressed as

$$E = I\left(R + j\omega L - \frac{j}{\omega C}\right) \tag{5.62}$$

The magnitude of the impedance Z of the circuit is given by Eqn. 5.62, and the phase angle by Eqn. 5.61. This mode of representation is very convenient for expressing both the magnitudes and phase relations of current and voltage.

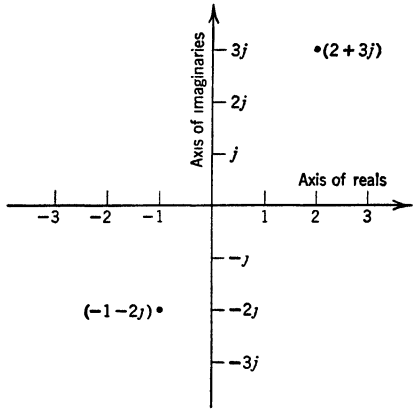


FIG. 5.15 Representation of complex quantities on a plane having axis of reals and axis of imaginaries.

b. The damping of a galvanometer. This is an example of the free oscillations of a galvanometer coil. In the ballistic galvanometer a quantity of electricity is discharged quickly through the coil of the galvanometer. A torque due to the electromagnetic forces is exerted on the coil which causes a deflection. If θ is the angular deflection of the coil at some instant of time, then the equation of motion of the coil at this instant is

$$I\frac{d^2\theta}{dt^2} = -b\frac{d\theta}{dt} - c\theta \tag{5.63}$$

In this expression $c\theta$ is the torque due to the twisting of the suspension of the coil through the angle θ and c is the torsion constant of the suspension, $b(d\theta/dt)$ is the torque due to the damping of the coil which has an angular velocity $d\theta/dt$ at the particular instant, $I\ d^2\theta/dt^2$ is the product of the moment of inertia of the coil about its axis of oscillation, and $d^2\theta/dt^2$ is its angular acceleration. This motion is damped simple harmonic motion as is seen by comparing Eqn. 5.63 with Eqn. 5.25. Just as in the linear motion of the spring, so in the angular motion of the coil the motion may be overdamped, critically damped, or damped oscillatory. Which of

these three types of motion results depends on whether b^2 is greater than, equal to, or less than $4cI$.

Let us suppose that the galvanometer is so constructed as to produce damped oscillatory motion. Then in the ballistic case the initial deflection is smaller than it would have been without any damping. The deflection without damping can be calculated from the initial observed deflection by means of the logarithmic decrement δ. The logarithmic decrement is the natural logarithm of the ratio of two maxima a period apart. In the linear case δ is given by Eqn. 5.33 as

$$\delta = \frac{RP}{2m}$$

and this corresponds in the galvanometer problem to

$$\delta = \frac{bP}{2I}$$

In general the quantities b and I cannot be calculated separately, but by observing successive maximum swings the value of δ may be obtained. Suppose then that δ is known for the galvanometer. If θ_1 is the initial observed deflection and since it takes a quarter of a period $P/4$ to go from the central undeflected position to θ_1, then the deflection θ_0, which would have occurred had there been no damping, is given approximately by

$$\log_e \frac{\theta_0}{\theta_1} = \frac{b}{2I} \cdot \frac{P}{4} = \frac{\delta}{4}$$

or

$$\theta_0 = \theta_1 e^{\delta/4} \tag{5.64}$$

c. Dispersion theory in light and x-rays. In the dispersion theory of light and x-rays it is assumed that the electric field of the electromagnetic wave falling on a substance acts on the electrons of the atoms of the substance. The electrons are displaced from their normal positions, causing the electromagnetic wave to be refracted. This refraction is measured in terms of the refractive index. The refractive index varies with the frequency of the electromagnetic wave, and it is the object of the dispersion theory to provide a relationship between the refractive index and the frequency.

Let us assume the electron to be a point charge having a mass of m gm and a charge of e electrostatic units. If an electromagnetic wave having an electric field intensity given by $E_0 \cos \omega t$ acts on the electron, then a force $E_0 e \cos \omega t$ is exerted on the electron. The electron is displaced a distance y from its equilibrium position, and owing to this displacement a restoring force ky is set up. Let us further assume that the motion of the electron is not damped. The problem then is one of forced oscillations without damping.

The equation of motion of the electron is similar to Eqn. 5.35 and is given by

$$my + ky = E_0 e \cos \omega t \tag{5.65}$$

In Eqn. 5.65 the frequency of the electromagnetic wave producing the forced oscillations of the electron is $f = \omega/2\pi$.

The solution of Eqn. 5.65 giving the displacement y at any time t is obtained for the steady state from Eqn. 5.41 as

$$y = \frac{F_0 \cos(\omega t - \theta)}{k - m\omega^2} = \frac{F_0 \cos(\omega t - \theta)}{m(\omega_0^2 - \omega^2)}$$

$$= \frac{eE_0 \cos(2\pi f t - \theta)}{4\pi^2 m(f_0^2 - f^2)} \tag{5.66}$$

where $\omega_0 = 2\pi f_0 = \sqrt{k/m}$ and f_0 is the natural frequency of the electron within the atom.

In the present example we are interested only in the amplitude of the oscillations. The displacement y of the electron from its equilibrium position separates the positive from the negative charge of the atom in such a manner that the atom becomes an electric dipole. This electric dipole has an electric moment given by the magnitude of either the positive or the negative charge multiplied by the distance between the charges. At time t when the separation is y the dipole moment of the atom is ey.

Now from electromagnetic theory, by a proof which would take us too far afield, it follows that the dielectric constant K of a gas having a refractive index n and N electrons of natural frequency f_0 per unit volume is

$$K = n^2 = 1 + \frac{4\pi Ney}{E_0}$$

so that from Eqn. 5.66, using the amplitude only, it follows that the square of the refractive index is

$$n^2 = 1 + \frac{Ne^2}{\pi m(f_0^2 - f^2)}$$

The phenomenon of anomolous dispersion in which the refractive index becomes very large is explained by this theory by assuming that in this region the frequency f of the electromagnetic waves becomes almost equal to the frequency f_0 of the electron in the atom. For many substances it appears that the frequency f_0 is much larger than the frequency f of the visible light waves, so that in this region we may write

$$n^2 = 1 + \frac{Ne^2}{\pi m f_0^2}$$

and, since the second term on the right is very small for gases, the refractive index n, by the binomial theorem, becomes

$$n = 1 + \frac{Ne^2}{2\pi m f_0^2}$$

PROBLEMS

1. A particle whose mass is 10 gm moves with undamped simple harmonic motion with a frequency of 5 per second. If the particle starts from rest 5 cm below its equilibrium position, find:

a. The displacement, velocity, and acceleration of the particle at any time t; plot these quantities on a graph over a time interval of one period.

b. The maximum kinetic and potential energies of the particle.

c. The velocity, acceleration, kinetic, and potential energies of the particle when it has a displacement of 3 cm. Show that the total energy of the particle is the same as that obtained in part *b*.

2. A particle is executing undamped simple harmonic motion with a frequency of 10 per second. Find the displacement y at any time t using the following initial conditions.

a. $t = 0; y = A; \dot{y} = 0.$
b. $t = 0; y = 0; \dot{y} = -v.$
c. $t = 0; y = 0; \dot{y} = +v.$
d. $t = 0; y = 3; \dot{y} = 5.$

3. The end of one of the prongs of a tuning fork which executes simple harmonic motion of frequency 200 per second has an amplitude of 0.5 mm. If damping is neglected, find:

a. The maximum velocity and acceleration of the end of the prong.

b. The velocity and acceleration of the end of the prong when it has a displacement of 0.25 mm.

4. A block is placed on the top of a piston which is executing simple harmonic motion in a vertical direction. The piston has an amplitude of 5 cm. Find:

a. At what position in the motion of the piston the block is likely to leave the piston.

b. The maximum frequency of the piston at which the block will remain continuously in contact with the piston at all times during the motion.

5. The piston of an automobile has a weight of 4 lbw and executes simple harmonic motion with a frequency of 20 per second and an amplitude of 2 in. Find the maximum velocity, acceleration, and force of the piston.

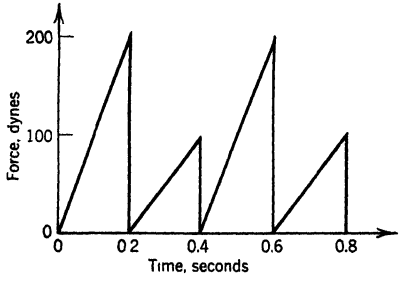
6. Suppose that a hole were bored along a diameter of the earth and a particle whose mass is 5 gm were dropped into it at the surface of the earth.

a. Show that the particle will execute simple harmonic motion and find its period.

b. Find the maximum kinetic energy of the particle. Take the radius of the earth as 6.37×10^8 cm and its mass as 6.1×10^{27} gm. Assume that the earth is of constant density.

7. A particle moves in a circle of radius r with a constant speed v. Show that the projection of this motion on any diameter is simple harmonic motion with a period of $2\pi r/v$.

8. A periodic force F given in dynes varies in a manner shown in the diagram. Find the average force acting over a period of 0.4 sec.



PROBLEM 8

9. A particle whose mass is 10 gm which is free to move in a plane is attracted towards a fixed point O with a force proportional to its distance from the fixed point. When its distance from the fixed point is 10 cm, the force of attraction is 10,000 dynes.

a. Find the period of the motion.

b. Suppose that the particle is started with a displacement of 10 cm from O in the X direction and a velocity of 20 cm per sec in the perpendicular Y direction. Find the equation for and plot the path of the particle.

c. Suppose that the particle is started with a displacement of 10 cm from O in the X direction and a velocity of 20 cm per sec in a direction towards the fixed point at an angle of 60° with the X axis. Find the equation of the path and plot this.

10. Electrons in an oscilloscope are deflected by two mutually perpendicular electric fields in such a manner that the displacement at any time t is given by

$$x = E_1 \sin \omega t \qquad y = E_2 \sin (\omega t + \delta)$$

a. If $E_1 = E_2 = 16$ cm and the phase angle δ is 90°, show that the path of the electrons is a circle whose radius is 16 cm

b. If $\delta = 0°$, show that the path is a straight line making an angle with the X axis whose tangent is E_2/E_1

c. If $\delta = 30°$, find the equation of the path and show that it is an ellipse.

11. A particle of mass m moving along the X axis is attracted towards the origin with a force of mkx. At a time t of 1 sec the particle is at the origin, and at time t of 2 sec the velocity of the particle is 2 cm per sec. If the period of the simple harmonic motion is 6 sec, then show that the force constant k is $\pi^2/9$ and the amplitude of the motion is $12/\pi$.

12. Lissajous' figures are produced by a particle's being subjected to two simple harmonic motions at right angles, along the X and Y axes respectively. Suppose that the frequency along the Y axis is twice that along the X axis and that the two simple harmonic motions have equal amplitudes A and a phase difference of 90° and are started with $x = A$ and $y = 0$. Show that the equation of the path is

$$y^2 = \frac{4x^2}{A^2}(A^2 - x^2)$$

and plot the path for an amplitude A of 4 cm.

13. Plot the path of a particle subjected to two simple harmonic motions at right angles to each other and of the same amplitude A, if the frequency along the X axis is $\frac{3}{4}$ of that along the Y axis and the motions are started in phase at $x = A$ and $y = A$.

14. If a pendulum clock has a period of exactly 1 sec when its amplitude is 4°, find the number of seconds gained or lost per day when the amplitude is 10°, assuming that the clock behaves as a simple pendulum. Find the maximum tension in the member holding the bob if the bob weighs 1 lb.

15. A simple pendulum consists of a weightless inextensible cord 100 cm in length and having a small mass of 50 gm placed on one end. If it is started from rest with an amplitude of 20° and there are no dissipative forces acting, find the velocity of the mass when the cord is:

a. Vertical.

b. At an angle of 5° with the vertical.

c. Find the tension in the cord for the condition in part *b*.

16. Show that by integration of Eqn. 5.18

$$ml\ddot{\theta} = -mg \sin \theta$$

the kinetic energy at any angular displacement θ is equal to the loss in potential energy at this displacement or that there is conservation of mechanical energy for the simple pendulum.

17. Show that the solution for critical damping, namely Eqn. 5.28 or $y = (A + Bt)e^{-Rt/2m}$, is a solution of the equation of motion, Eqn. 5.25.

18. Show that the logarithmic decrement δ, obtained by taking two successive maximum displacements on the same side of the equilibrium position for the data in Fig. 5.7*c*, is about 1.28.

19. Show that the logarithmic decrement δ equal to $RP/2m$ has no dimensions.

20. Find the time for which the kinetic energy of the moving mass of 25 gm in section 5.7, Fig. 5.7*a*, is a maximum for the overdamped motion. Plot a graph of the kinetic energy of the mass against time for the overdamped motion.

21. Repeat problem 20 for the critically damped system, Fig. 5.7*b*.

22. Repeat problem 20 for the damped oscillatory system, Fig. 5.7*c*. Find the energy lost between the displacements of −5.0 cm and zero, and between zero and 2.63 cm. Give a physical explanation for the kinetic energy's not having its maximum value at zero displacement.

23. Show that the work done against the damping force during the half-cycle from the times 0.181 to 0.501 sec in Fig. 5.7*c* is equal to the change in the kinetic energy during this interval, namely about 11,000 ergs. (The necessary integrals can be found in Pierce's Table of Integrals, numbers 935, 937, and 938.)

24. Calculate the loss in energy for the mass executing damped oscillations as shown in Fig. 5.7*c* in the half-vibration between the times 0.501 and 0.822 sec.

25. Find the equation for the displacement with time for the forced oscillation using the data employed in Fig. 5.8 except for the damping force constant which is now $R = 1000$ dyne sec per cm. Plot a curve of the displacement against time.

26. A body weighing 2 lb is hung on the lower end of a spring and produces an extension of 0.5 in. Assuming no damping, find the period of the simple harmonic motion.

27. A force of 2500 dynes produces a stretch of 1 cm in a spring. If a mass of 20 gm is hung on the lower end of the spring and the mass is pulled down a distance of 5 cm below the equilibrium position and then let go so as to move without damping, find:

a. The period of the simple harmonic motion.

b. The maximum energy of the mass.

c. The maximum velocity of the mass.

d. How the total mechanical energy of the system varies with time.

28. If the spring in problem 27 is subjected to a damping force of 1000 dyne sec per cm with the initial displacement of −5 cm, find:

a. The displacement of the spring after 0.1, 0.5, and 2.0 sec.

b. The initial energy of the system.

c. The energy after 0.1 sec.

29. If the spring in problem 27 is subjected to a damping force of 200 dyne sec per cm with the initial displacement of −5 cm, find:

a. The equation of the displacement y at any time t.

b. The period of the damped oscillations.

c. The logarithmic decrement from two maximum displacements.

d. The logarithmic decrement from the equation $RP/2m$.

e. The loss in energy in the first complete cycle.

30. A weight of 20 lb is hung on a spring whose restoring force is 40 lbw per ft. The spring is started from rest with an upward displacement of 0.25 ft. Find the equations for the displacement y at any time t and plot the curves, using time intervals of 0.2 sec in the range from zero to 1.6 sec under the following conditions:

a. Zero damping force.

b. A damping force of 1.5 lbw sec per ft.

c. A damping force of 25 lbw sec per ft.

d. The critical damping force.

31. The spring in problem 30 is subjected to a periodic driving force of 10 cos ωt lbw and a damping force of 1.5 lbw sec per ft. For the steady-state condition find the equations for the displacement y at any time t with the angular velocity ω equal to:

a. $\sqrt{k/m}$, the natural angular velocity without damping.

b. $\sqrt{k/m - R^2/2m^2}$, the angular velocity for displacement resonance.

c. $\sqrt{k/m - R^2/4m^2}$, the natural angular velocity of the free system with damping.

32 An electric charge sent through a ballistic galvanometer produces an initial deflection of 5.0 cm to the right of the zero mark of a scale. Subsequent deflections on alternate sides of the zero mark are -4.80, 4.61, -4.42, 4.25. From this data find:

a. The logarithmic decrement.

b. The initial deflection which would have occurred if there had been no damping.

33. An electrical circuit is composed of a resistance of 40 ohms, a condenser having a capacity of 100 microfarads, and an inductance of 0.2 henry connected in series to an alternating voltage of 150 sin ($120\pi t$) volts.

a. Find the maximum value of the current in the circuit.

b. Find the root-mean-square current in the circuit.

c. If the capacitance of the condenser can be varied, find what value it must have if the current is to be a maximum. Find the value of this current.

6 · TRANSLATIONAL AND ROTATIONAL MOTION OF RIGID BODIES

6.1 Internal Forces in Rigid Bodies

Up to the present we have been concerned with the motion of particles or of bodies that can be treated as particles, where no rotational motion is involved. In the present chapter we shall extend the concepts introduced earlier to bodies of finite size that can possess both rotational motion and translational motion. In dealing with these bodies we shall assume them to be rigid; that is, they do not change size or shape under the action of forces. Although this assumption may not be strictly accurate, it is sufficiently so for our present purposes.

The rigidity of a body is due to the nature of the internal attractive forces between the molecules composing the body. We shall assume that the internal force between any two molecules acts along the line joining them and that Newton's third law is applicable. Thus for any two molecules or particles i and j in a body it is assumed that the internal force $\mathbf{F}_{ij}$ exerted on i by j is equal and opposite to the force $\mathbf{F}_{ji}$ exerted on j by i. In symbols

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

Therefore the sum of the internal forces within any body is zero.

6.2 Translational Motion and Center of Mass

In this section we shall show that for linear or translational motion the total external force acting on the particles composing a body is equal to the summation of the time rate of change of momentum of the particles. Also the total external force is equal to the total mass of the particles multiplied by the acceleration of their center of mass.

To prove this let us consider a body made up of a number of particles whose masses are m_1, m_2, $\cdots$, m_i, etc., respectively, and whose positions relative to some fixed origin O in an inertial system are given by the vector distances $\mathbf{r}_1$, $\mathbf{r}_2$, $\cdots$, $\mathbf{r}_i$, etc., as shown in Fig. 6.1. Let the external force acting on the particle of mass m_i be $\mathbf{F}_i$

and the corresponding internal force be $\mathbf{F}_i'$. By Newton's second law

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{F}_i'$$

Taking the summation for all the n particles of the body, we have

$$\sum_1^n m_i\ddot{\mathbf{r}}_i = \sum_1^n \mathbf{F}_i + \sum_1^n \mathbf{F}_i'$$

where the symbol Σ denotes a summation. Now the internal forces consist of equal and opposite pairs, and for the body as a whole the sum of these forces, $\sum_1^n \mathbf{F}_i'$, is zero. Thus

$$\sum_1^n m_i\ddot{\mathbf{r}}_i = \sum_1^n \frac{d^2}{dt^2}(m_i\mathbf{r}_i) = \sum_1^n \mathbf{F}_i = \mathbf{F} \tag{6.1}$$

where $\mathbf{F}$ is the resultant external force acting on the whole of the particles.

An important development of this theory is brought about by the introduction of the concept of *center of mass*. The center of mass,

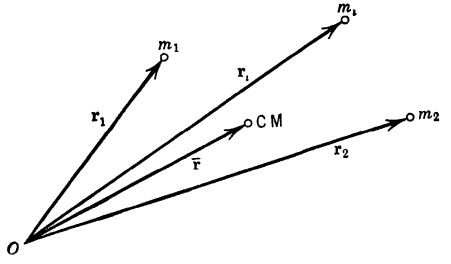


FIG. 6.1 Center of mass of a number of particles.

C.M., of a body or of a system of particles is that point in the body or system of particles at which, for purposes of calculation, all the mass may be assumed to be concentrated. Relative to the origin O, the vector position $\bar{\mathbf{r}}$ of the center of mass of the n particles in Fig. 6.1 is defined as

$$\bar{\mathbf{r}} = \frac{\sum_1^n m_i\mathbf{r}_i}{\sum_1^n m_i} = \frac{\sum_1^n m_i\mathbf{r}_i}{M} \tag{6.2}$$

where M is the total mass, $\sum_1^n m_i$, of all the particles. By differentiation of Eqn. 6.2 we have

$$M\ddot{\mathbf{r}} = \sum_1^n m_i \ddot{\mathbf{r}}_i$$

and from Eqn. 6.1 it follows that

$$M\ddot{\mathbf{r}} = \mathbf{F} \tag{6.3}$$

This is an important result showing that *for a group of particles or a rigid body the translational motion of the center of mass behaves as though all the mass of the particles were concentrated at the center of mass and the resultant external force also acted at that point.* In other words the motion of the center of mass of a body is the same as that of a particle having the total mass of the body and the same resultant external force acting on it. This result applies either to a system of particles in motion, as the molecules of a gas, or to a rigid body. In all cases the resultant internal force must be zero if Eqn. 6.3 is to be applied.

If a hammer or other body is thrown at some angle with the vertical, then its center of mass will follow a parabolic path just as would a particle with the same initial motion. This applies whether the hammer has an initial rotational motion or not. The resultant external force is the weight of the hammer, and the acceleration of the center of mass is the acceleration due to gravity. On a larger scale, if we assume that there is no resultant external force acting on the solar system as a whole, then its center of mass will either be at rest or move with constant speed in a straight line.

For many purposes of calculation it is convenient to express the position of the center of mass in terms of its three Cartesian coordinate components. Thus, from the vector position $\bar{\mathbf{r}}$ of the center of mass given by Eqn. 6.2, three equations may be written giving the X, Y, Z coordinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, respectively, of the center of mass. These are

$$\bar{x} = \frac{\sum_1^n m_i x_i}{M} \qquad \bar{y} = \frac{\sum_1^n m_i y_i}{M} \qquad \bar{z} = \frac{\sum_1^n m_i z_i}{M} \tag{6.4}$$

where x_i, y_i, z_i are the coordinates of the particle m_i. The vector $\mathbf{r}_i$ giving the position of m_i may be written

$$\mathbf{r}_i = \mathbf{i}x_i + \mathbf{j}y_i + \mathbf{k}z_i$$

and the vector $\bar{\mathbf{r}}$ giving the position of the center of mass

$$\bar{\mathbf{r}} = \mathbf{i}\bar{x} + \mathbf{j}\bar{y} + \mathbf{k}\bar{z}$$

If F_x, F_y, F_z are the components of the external force $\mathbf{F}$, then Eqn. 6.3 may be written

$$F_x = M\ddot{\bar{x}} \qquad F_y = M\ddot{\bar{y}} \qquad F_z = M\ddot{\bar{z}} \tag{6.5}$$

Thus the component of the external force in the X direction is equal to the product of the total mass and the component of the acceleration of the center of mass in the X direction.

Example of translational motion of the center of mass of a number of particles. Consider that four masses of 30 gm, 50 gm, 20 gm, and 100 gm are placed in the

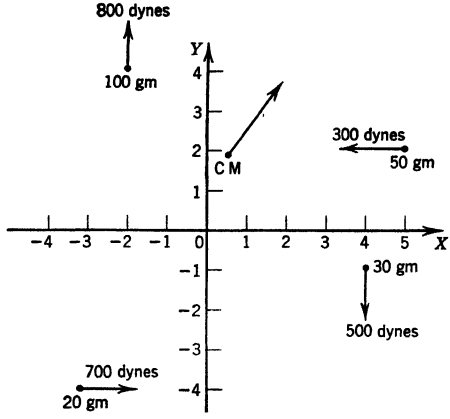


FIG. 6.2 Motion of the center of mass of four particles.

XY plane and have coordinates of (4, −1), (5, 2), (−3, −4), and (−2, 4) respectively. These are shown in Fig. 6.2 with the external forces acting on them. Let us first determine the coordinates of the center of mass. From Eqn. 6.4

$$\bar{x} = \frac{\sum_1^4 m_i x_i}{M}$$

$$= \frac{(30 \times 4) + (50 \times 5) + (20 \times -3) + (100 \times -2)}{200}$$

$$= \tfrac{110}{200} = 0.55$$

Similarly

$$\bar{y} = (30 \times -1) + (50 \times 2) + (20 \times -4) + (100 \times 4)$$

$$= \tfrac{390}{200} = 1.95$$

Thus the coordinates of the center of mass of the particles are (0.55, 1.95).

Suppose that the external forces act on the masses as shown in Fig. 6.2. The problem is to calculate the acceleration of the center of mass. In the X direction the resultant force is

$$F_x = 700 - 300 = 400 \text{ dynes}$$

and in the Y direction

$$F_y = 800 - 500 = 300 \text{ dynes}$$

The resultant force on the particles has a magnitude of

$$F = \sqrt{400^2 + 300^2} = 500 \text{ dynes}$$

and a direction that makes an angle θ with the X axis given by

$$\tan \theta = \tfrac{300}{400} = \tfrac{3}{4}$$

From Eqn. 6.5 it follows that the acceleration of the center of mass is

$$a = \frac{F}{M} = \frac{500}{200} = 2.5 \text{ cm per sec per sec}$$

and it makes an angle with the X axis whose tangent is $\frac{3}{4}$. Thus, while the particles would change their relative positions with time, their center of mass would proceed with the above acceleration.

6.3 The Center of Mass of a Portion of a Body

It is readily proved that, if the center of mass of the whole of a body and that of a portion of the body are known, then the center of mass of

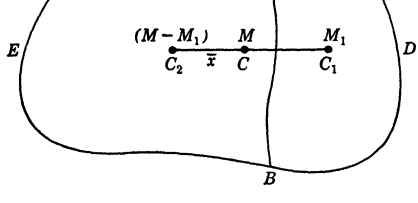


FIG. 6.3 Center of mass of a portion of a body.

the remainder may be found. Let C be the center of mass of the whole body $ADBEA$, and C_1 that of the portion $ADBA$, see Fig. 6.3, where $ADBA$ may be considered to be cut out of the whole body. The problem is to find the center of mass C_2 of the remaining portion $ABEA$, given the positions C and C_1 and the corresponding masses of these portions.

Let M be the mass of the whole body $ADBEA$ and M_1 that of the

portion $ADBA$. The center of mass C of the whole body may be considered the center of mass of two point masses, M_1 at C_1 and $(M - M_1)$ at C_2, where C_2 lies on the line C_1C. Let C_2 be the origin of the X axis; then from Eqn. 6.4

$$\bar{x} = C_2C = \frac{\sum_1^n m_i x_i}{M} = \frac{M_1}{M} C_1C_2 = \frac{M_1}{M} (C_1C + CC_2)$$

or

$$C_2C = \frac{M_1}{M - M_1} C_1C$$

Thus when a known portion is cut away from a body, if the centers of mass of the whole and the portion are known, the center of mass of the remainder can be found.

6.4 The Centers of Mass of Various Symmetrical Bodies

a. Center of Mass of a Uniform Rectangle. The problem is to find the center of mass of a uniform rectangle whose length is a, whose width is b, and whose mass is M. We shall prove what is more or less

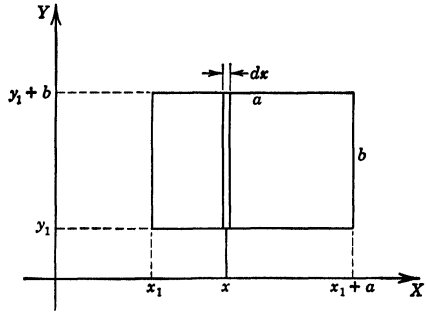


FIG. 6.4 Center of mass of a uniform rectangle.

obvious, that the center of mass is at the geometrical center of the rectangle.

In a uniformly distributed mass the summation in the expression for the center of mass is replaced by an integration. Thus the equivalent equations replacing Eqn. 6.4 are

$$\bar{x} = \frac{\int x\, dm}{M} \qquad \bar{y} = \frac{\int y\, dm}{M} \qquad \bar{z} = \frac{\int z\, dm}{M} \tag{6.6}$$

where the integrations are carried out over the whole of the body of mass M. Suppose that the rectangle lies in the XY plane with the lower left-hand edge having the coordinates x_1, y_1 and placed as shown in Fig. 6.4. To find the X coordinate $\bar{x}$ of the center of mass, consider a narrow strip at position x whose width is dx and whose height is b. If the mass of the rectangle of area ab is M, then the mass of the strip is

$$dm = b\,dx\,\frac{M}{ab}$$

Then from Eqn. 6.6

$$\bar{x} = \frac{1}{M}\int_{x_1}^{x_1+a} \frac{M}{ab}(b\,dx)x = \frac{1}{a}\left[\frac{x^2}{2}\right]_{x_1}^{x_1+a}$$
$$= x_1 + \frac{a}{2}$$

Similarly the Y coordinate of the center of mass $\bar{y}$ is given by

$$\bar{y} = y_1 + \frac{b}{2}$$

Thus the center of mass has the coordinates $a/2$, $b/2$ relative to the lower left-hand edge; that is, the center of mass is at the geometrical center of the rectangle.

b. Center of Mass of a Uniform Right Circular Solid Cone. Suppose that the cone has a vertical height h and a base with a radius a. For convenience we shall place the vertex of the cone at the origin of the coordinate system and the axis of symmetry of the cone along the Y axis as in Fig. 6.5. Consider a horizontal thin slab of the cone whose thickness is dy, whose mean radius is r, and whose height above the origin is y. If the density or mass per unit volume of the cone is ρ, then the mass dm of the thin slab is

$$dm = \pi r^2 \rho\,dy$$

The radius r of the slab is proportional to its height y, and from the figure it can be seen that

$$\frac{r}{a} = \frac{y}{h} \qquad \text{or} \qquad r = \frac{ay}{h}$$

Thus

$$dm = \frac{\pi\rho a^2 y^2}{h^2}\,dy$$

and the total mass M of the cone is

$$M = \frac{\pi\rho a^2}{h^2}\int_0^h y^2\,dy = \frac{\pi\rho a^2 h}{3}$$

The vertical distance of the center of mass from O is

$$\bar{y} = \int_0^h \frac{y\,dm}{M}$$

$$= \frac{\pi\rho a^2}{Mh^2}\int_0^h y^3\,dy$$

Integrating and substituting the value for M, we have

$$y = \frac{3\pi\rho a^2 h^4}{4h^2\pi\rho a^2 h} = \frac{3h}{4}$$

or the center of mass is $\frac{3}{4}$ of the altitude measured from the vertex.

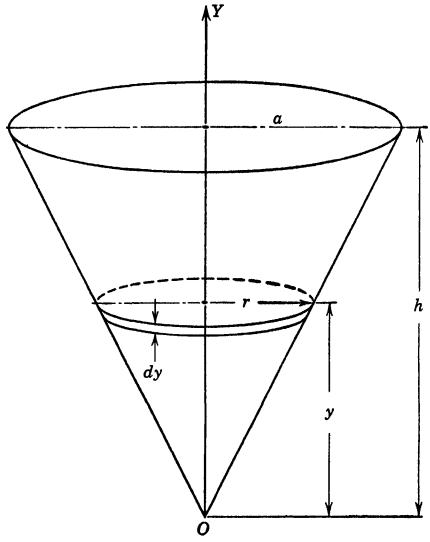


FIG. 6.5 Center of mass of a cone.

It can be readily appreciated that the center of mass can be calculated only for bodies having some degree of symmetry. For such

bodies one must use judgment as to how to place the axes relative to an axis or plane of symmetry. With a little care, considerable time and effort may be saved.

6.5 Experimental Determination of the Center of Mass of an Unsymmetrical Body

In unsymmetrical bodies the center of mass cannot be calculated simply. It must be obtained experimentally. The principle of such an experiment may be illustrated for a plane object as shown in Fig. 6.6. A smooth peg passing through a small hole as at O is used to suspend the object. A piece of string with a weight on the end is fastened to the peg at O to indicate the vertical direction. The line OA in the direction of the string is drawn on the object. Now in principle it is the center of weight or *center of gravity*, C.G., which lies on the line OA. Each element of mass dm has a weight $g\,dm$ where g is the acceleration due to gravity. If we define the center of gravity of a body as the point in the body at which all the weight may be assumed to be concentrated, then the center of gravity of the plane in Fig. 6.6 must lie somewhere along the line OA. Similarly another line $O'A'$ may be drawn on which the center of gravity must lie. The intersection of these lines gives the position of the center of gravity of the body.

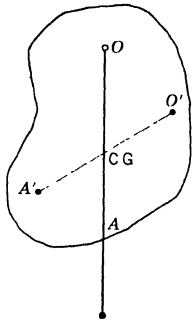


FIG. 6.6 Experimental determination of center of mass.

We may now ask whether the position of the center of gravity is also the position of the center of mass? The equations giving the position of the center of gravity relative to the X, Y, Z axes are similar to Eqns. 6.6 for the center of mass. The x coordinate of the center of gravity is defined as

$$\bar{x} = \frac{\int gx\,dm}{\int g\,dm} \tag{6.7}$$

If the acceleration due to gravity g is assumed to be the same in magnitude and direction at all points on the body, then the center of gravity coincides with the center of mass. This can be assumed to be correct for all ordinary objects. In dealing with the center of gravity we are finding the resultant of a series of forces. If g is not considered constant in direction and magnitude throughout the body, then the forces are neither parallel to each other nor proportional to the elements of mass, and almost impossible mathematical equations result.

6.6 Rotation of a Uniform Symmetrical Disk about a Fixed Axis through Its Center

In this section we shall prove in an elementary manner that for rotational motion about a fixed axis the total moment of the forces or torques about this fixed axis is equal to the time rate of change of the angular momentum of the body about the fixed axis. This rotational motion introduces a new concept called the moment of inertia or the

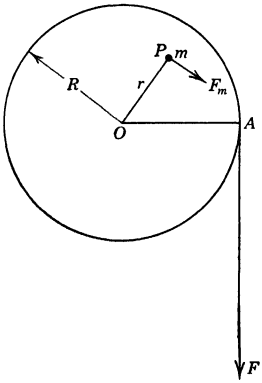


FIG. 6.7 Rotation of a disk about a fixed axis.

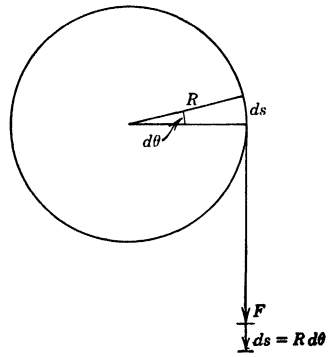


FIG. 6.8 Kinetic energy of a rotating disk.

rotational inertia of a body. In terms of this the torque about a fixed axis is equal to the product of the moment of inertia and the angular acceleration of the body about the axis.

Consider a plane circular disk free to rotate about a fixed axis through its center O and perpendicular to its plane. A light cord is coiled around the periphery of the disk, and a constant vertical force F is applied as shown in Fig. 6.7. (Note that this is not equivalent to replacing the constant force F with a hanging weight when the system is moving.) As the cord is pulled downward by the force F, the disk rotates with an increasing angular velocity. Suppose that the angular velocity at some instant of time is ω radians per second. Then its angular acceleration α is the time rate of change of this angular velocity:

$$\alpha = \frac{d\omega}{dt}$$

This angular acceleration depends on both the force F and the perpen-

dicular distance of the force F from the axis of rotation. The product of the force F and this perpendicular distance is called the *moment of the force* F, or *the torque* L, about O. From Fig. 6.7 the torque L for the force F about O is

$$L = F \times OA = FR$$

where R is the radius of the disk.

Consider now a particle of mass m in the disk at a distance OP or r from the axis of rotation. If at the instant considered the angular velocity of the disk is ω, then the tangential velocity of the mass m in a direction perpendicular to OP is $r\omega$. The instantaneous linear momentum of the mass m is $mr\omega$, and the *moment of this momentum* or the *angular momentum* about the axis of rotation is $mr^2\omega$. For all the particles of the disk the total moment of momentum Ω of the disk about O is

$$\Omega = \Sigma mr^2\omega = \omega\Sigma mr^2$$

where ω is the same for all the particles in the disk and the summation is taken over every particle in the disk. The quantity Σmr^2 plays a very important role in rotational motion and is called the *moment of inertia* or the *rotational inertia* I of the body about the axis of rotation. Thus we may write

$$\Omega = I\omega \tag{6.8}$$

To obtain the relationship between the torque L and the angular acceleration α, consider the force F_m acting on the particle m at P. From Newton's second law this is

$$F_m = mr\alpha = mr\frac{d\omega}{dt}$$

and the moment of this force about the axis of rotation is

$$F_m r = mr^2\alpha = mr^2\frac{d\omega}{dt}$$

For the whole disk the total moment of the forces acting on all the particles of the disk must be equal to the moment of the applied force L. Thus

$$L = FR = \sum mr^2\alpha = \sum mr^2\frac{d\omega}{dt}$$

$$= I\alpha = I\frac{d\omega}{dt}$$

From Eqn. 6.8 it follows that the time rate of change of the angular

momentum $\dot{\Omega}$ is equal to the product of the moment of inertia I and the rate of change of the angular velocity. Thus we may write for the rotation of a body about a fixed axis

$$L = I\alpha = \dot{\Omega} \tag{6.9}$$

Thus the torque about a fixed axis is equal to the product of the moment of inertia and the angular acceleration about the fixed axis.

6.7 Work and Rotational Kinetic Energy

We shall now determine the relationship between the work done on the disk by the force F and the increase in the kinetic energy of the disk. If at some instant of time the angular velocity of the disk is ω, then the disk will turn through an infinitesimal angle $d\theta$ in an infinitesimal time dt where

$$\omega = \frac{d\theta}{dt} \qquad \text{or} \qquad d\theta = \omega\, dt$$

During this time dt a point on the periphery of the disk moves through a distance ds given by

$$ds = R\, d\theta$$

The work done by the force F in this time is $F\, ds$, Fig 6.8. This work can be expressed as

$$F\, ds = FR\, d\theta = L\, d\theta$$

Thus the work done is equal to the product of the torque and the angle through which the disk turns. By substitution it follows that

$$L\, d\theta = I \frac{d\omega}{dt}\, \omega\, dt = I\omega\, d\omega$$

where $d\omega$ is the change in the angular velocity during the time dt. Suppose that the disk has an angular velocity ω_0 when the force is applied and that after the disk has turned through an angle θ the angular velocity has become ω. With these boundary conditions we have

$$\int_0^\theta L\, d\theta = \int_{\omega_0}^{\omega} I\omega\, dw$$

or

$$L\theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 \tag{6.10}$$

where $\frac{1}{2}I\omega^2$ is called the *angular or rotational kinetic energy* of a body having a moment of inertia I and an angular velocity ω. Thus the work done by the torque in turning the disk about a fixed axis through some angle is equal to the change in the angular or rotational kinetic

energy of the disk. The quantity $\frac{1}{2}I\omega^2$ plays the same role in rotational motion that $\frac{1}{2}mv^2$ does in linear motion. Equation 6.10 expresses the law of conservation of mechanical energy as applied to rotational motion.

In order to emphasize the ideas presented above we shall take a numerical example. Suppose that the disk shown in Fig. 6.7 has a mass of 500 gm and a radius of 10 cm. A constant force of 20,000 dynes is applied at the end of the cord. If the disk has an initial angular velocity of 4 radians per second when the force is applied, the problem is to find the angular velocity and rotational kinetic energy after 2 sec.

First we must calculate the value of the moment of inertia I of the circular disk about its axis of rotation. The definition given for the moment of inertia is

$$I = \Sigma mr^2$$

where the summation is taken over all the particles composing the body. For a continuously distributed mass such as a uniform solid body, for example the disk, one replaces the summation Σmr^2 used for a discrete series of masses by an integral. Thus for a uniform solid the moment of inertia is given by

$$I = \int r^2\, dm \tag{6.11}$$

The calculation of I for the disk about an axis through its center perpendicular to its plane can be made by considering the disk as made up of a large number of concentric circular rings. One of these having a radius of r and a width of dr is shown in Fig. 6.9. If the mass of the disk of radius R is M, then the mass of the ring of radius r and width dr is

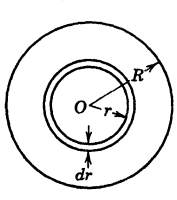


FIG. 6.9 Calculation of the moment of inertia of a circular disk.

$$dm = \frac{M}{\pi R^2} 2\pi r\, dr = \frac{2M}{R^2} r\, dr$$

Since, to a first approximation, all points on this ring lie at the same distance from the center, the moment of inertia of this ring about an axis perpendicular to its plane and through its center is

$$r^2\, dm = \frac{2M}{R^2} r^3\, dr$$

Thus I for the whole disk is

$$I = \int r^2\, dm = \int_0^R \frac{2M}{R^2} r^3\, dr = \frac{MR^2}{2} \tag{6.12}$$

In our problem $M = 500$ gm and $R = 10$ cm so that

$$I = \frac{MR^2}{2} = 2.5 \times 10^4 \text{ gm cm}^2$$

The torque L exerted on the disk is

$$L = FR = 2 \times 10^5 \text{ dyne cm}$$

From $L = I\alpha$ the angular acceleration of the disk α is

$$\alpha = \frac{2 \times 10^5 \text{ dyne cm}}{2.5 \times 10^4 \text{ gm cm}^2} = 8 \text{ radians per sec}^2$$

If the angular velocity at zero time is ω_0 and at time t is ω, then since $\alpha = d\omega/dt$ it follows that

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha \, dt$$

or

$$\omega - \omega_0 = \alpha t$$

Since the initial angular velocity ω_0 in this problem is 4 radians per second, then the angular velocity ω after 2 sec is

$$\omega = \omega_0 + \alpha t$$

$$= 4 + (8 \times 2) = 20 \text{ radians per second}$$

The rotational kinetic energy after 2 sec is

$$\tfrac{1}{2} I\omega^2 = \tfrac{1}{2} \times 2.5 \times 10^4 \times 400 = 5 \times 10^6 \text{ ergs}$$

It might be profitable to show numerically that Eqn 6 10 dealing with both work and kinetic energy is valid. We must first find the angle θ through which the disk turns in the 2 sec. Since $\omega = d\theta/dt$, it follows that, if $\theta = 0$ at $t = 0$, then

$$\int_0^{\theta} d\theta = \int_0^t \omega \, dt = \int_0^t (\omega_0 + \alpha t) \, dt$$

or

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$

In our example

$$\theta = (4 \times 2) + \tfrac{1}{2}(8 \times 4) = 24 \text{ radians}$$

so that the work done by the force in the 2 sec is:

$$L\theta = (2 \times 10^5 \times 24) = 4.8 \times 10^6 \text{ ergs}$$

This quantity should be equal to the change in kinetic energy during the 2 sec or

$$L\theta = \tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^2$$

$$= (5 \times 10^6) - (2 \times 10^5) = 4.8 \times 10^6 \text{ ergs}$$

which of course agrees with the value of $L\theta$ given above.

6.8 Two Theorems Relating to Moments of Inertia

Theorem a. *The moment of inertia of a uniform plane body about an axis perpendicular to its plane is equal to the sum of its moments of*

inertia about any two perpendicular axes in the plane that intersect on the first axis.

Let the plane object lie in the XY plane and let the perpendicular axis be the Z axis. From Fig. 6.10 the moment of inertia I_x about the X axis is given by

$$I_x = \int y^2 \, dm$$

and similarly that about the Y axis is

$$I_y = \int x^2 \, dm$$

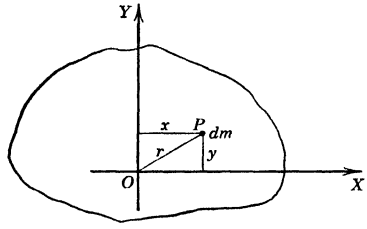


FIG. 6.10 Moments of inertia about perpendicular axes.

For any point P in the plane having coordinates x, y the distance r of P from O is

$$r^2 = x^2 + y^2$$

The moment of inertia I of the plane about the Z axis is

$$I_z = \int r^2 \, dm = \int (x^2 + y^2) \, dm$$

Thus

$$I_z = I_x + I_y \tag{6.13}$$

As an example of the application of this theorem let us find the moment of inertia of a plane circular disk about any diameter using Eqn. 6.12. If the disk lies in the XY plane with its center at the origin of the coordinates, then

$$\frac{MR^2}{2} = I = I_x + I_y$$

For the circular disk

$$I_x = I_y$$

Hence the moment of inertia of a plane circular disk about any diameter is

$$I_x = I_y = \frac{MR^2}{4}$$

Theorem b, or the theorem of parallel axes. *The moment of inertia I of a body about any axis is equal to the moment of inertia $I_{C.M.}$ of the*

body about a parallel axis through its center of mass plus the mass M of the body times the square of the perpendicular distance l between the axes:

$$I = I_{\text{C M.}} + Ml^2$$

For convenience let the axis about which the moment of inertia is to be taken be the Z axis which is perpendicular to the XY plane. Let the coordinates of the center of mass be $\bar{x}$, $\bar{y}$, $\bar{z}$. If a particle at

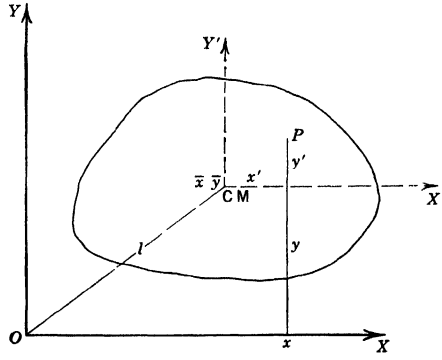


FIG. 6.11 Moment of inertia about parallel axes.

P in Fig. 6.11 of mass dm has coordinates x, y, z relative to the X, Y, Z axes and x', y', z' relative to a set of parallel axes X', Y', Z' through the center of mass of the body, then

$$x = x' + \bar{x} \qquad \text{and} \qquad y = y' + \bar{y}$$

The moment of inertia I of the body about the OZ axis is

$$I = \int (x^2 + y^2)\, dm = \int [(x' + \bar{x})^2 + (y' + \bar{y})^2]\, dm$$
$$= \int (x'^2 + y'^2)\, dm + (\bar{x}^2 + \bar{y}^2) \int dm + 2\bar{x} \int x'\, dm + 2\bar{y} \int y'\, dm$$

Since the coordinates x', y' are chosen with respect to axes through the center of mass, then by definition of the center of mass

$$\int x'\, dm = 0 \qquad \text{and} \qquad \int y'\, dm = 0$$

Let the moment of inertia about the Z' axis through the center of mass be denoted by $I_{\text{C.M.}}$, and let the distance between the Z and Z' axes be l so that

$$I_{\text{C.M}} = \int (x'^2 + y'^2)\, dm$$

and

$$l^2 = (\bar{x}^2 + \bar{y}^2)$$

Then

$$I = I_{\text{C.M.}} + Ml^2 \tag{6.14}$$

As an example of the application of this theorem we can find the moment of inertia of a plane circular disk about an axis at its edge perpendicular to its plane. For the axes through the center of the disk and perpendicular to its plane

$$I_{\text{C.M}} = \frac{MR^2}{2}$$

Since we have assumed the circular disk to be uniform, the center of mass must be at its geometrical center. Hence for a parallel axis through the edge of the disk at a distance R from the center of mass the moment of inertia I is

$$I = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}$$

6.9 Radius of Gyration

The radius of gyration k of a body of mass M and of moment of inertia I is defined as

$$I = Mk^2 \tag{6.15}$$

Thus we see that the moment of inertia I of a body of mass M and radius of gyration k is the same as that of a particle of mass M placed at a distance k from the axis of rotation. Notice that the radius of gyration applies to a particular axis. For the plane circular disk and for an axis perpendicular to its plane through its center of mass the radius of gyration is

$$k = \frac{R}{\sqrt{2}}$$

and for an axis parallel to its plane through the edge of the disk the radius of gyration is

$$k = \sqrt{\tfrac{3}{2}}\, R$$

6.10 Calculation of Moments of Inertia

1. Moment of Inertia of a Narrow Uniform Rod about an Axis through Its Center and Perpendicular to Its Length. Let the mass of the rod be M and its length be l. Then the mass per unit length of the rod σ is

$$\sigma = \frac{M}{l}$$

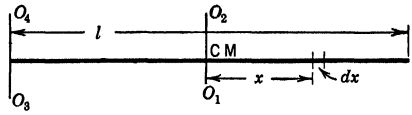


FIG 6.12 Moment of inertia of a uniform rod.

If the X axis lies along the length of the rod with its origin at the center of the rod, then the moment of inertia about O_1O_2 in Fig. 6.12 is

$$I_{C\,M} = \int_{-l/2}^{+l/2} x^2\, dm = \int_{-l/2}^{+l/2} x^2\sigma\, dx = \frac{\sigma l^3}{12} = \frac{Ml^2}{12}$$

The moment of inertia about an axis O_3O_4 parallel to O_1O_2 is

$$I = I_{C.M} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

2. Moment of Inertia of a Uniform Plane Rectangle about an Axis through Its Geometrical Center Perpendicular to Its Plane. Let the

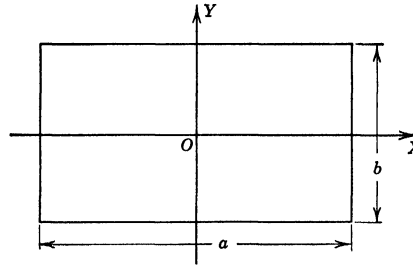


FIG. 6.13 Moment of inertia of a rectangle.

length of the rectangle be a, its width b, and its mass M, as in Fig. 6.13. Let us consider the rectangle as made up of narrow strips parallel to the Y axis. From the previous example the moment of inertia I_x about the X axis is

$$I_x = \frac{Mb^2}{12}$$

Similarly

$$I_y = \frac{Ma^2}{12}$$

From theorem a given above, the moment of inertia I about an axis

through the center O and perpendicular to the plane of the rectangle is given by

$$I = I_x + I_y$$

$$= \frac{M(a^2 + b^2)}{12}$$

3. Moment of Inertia of a Solid Uniform Sphere about Any Diameter. Let the sphere have a radius a and a mass M. To find the moment of inertia about a diameter, consider the sphere as made up

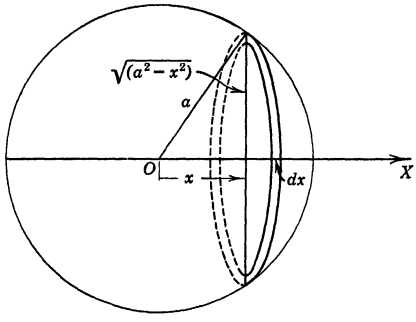


FIG. 6.14 Moment of inertia of a sphere about a diameter.

of a series of disks, one of which is at a distance x from the center and has a thickness dx, Fig. 6.14. The area of the disk is $\pi(a^2 - x^2)$ and, if the density of the sphere is ρ, the mass of the disk is $\pi\rho(a^2 - x^2)\,dx$. From Eqn. 6.12 the moment of inertia of the circular disk about the axis OX is

$$dI = \pi\rho(a^2 - x^2)\,dx\,\frac{(a^2 - x^2)}{2}$$

Hence the moment of inertia of the sphere is

$$I = \int_{-a}^{+a} \frac{\pi\rho}{2}(a^2 - x^2)^2\,dx$$

$$= \frac{\pi\rho}{2}\left[a^4x + \frac{x^5}{5} - \frac{2a^2x^3}{3}\right]_{-a}^{+a} = \frac{8\rho\pi}{15}a^5$$

Since the mass of the sphere is $M = 4\pi a^3\rho/3$, then

$$I = \tfrac{2}{5}Ma^2$$

From these examples it can be seen that it would not be possible to calculate the moment of inertia of an unsymmetrical body about any

axis. For such bodies the moments of inertia must be obtained experimentally, as for example by using the torsion pendulum, section 7.12. A table of moments of inertia of some common symmetrical objects is given below

TABLE OF MOMENTS OF INERTIA
(m is mass of body)

Body	Axis	Moment of Inertia
Uniform thin rod length l	Normal to length through center	$\frac{ml^2}{12}$
Uniform thin rod length l	Normal to length at one end	$\frac{ml^2}{3}$
Thin rectangular sheet, sides a and b	Through center parallel to side b	$\frac{ma^2}{12}$
Thin rectangular sheet, sides a and b	Through center perpendicular to sheet	$\frac{m(a^2+b^2)}{12}$
Thin circular disk, radius r	Through center perpendicular to disk	$\frac{mr^2}{2}$
Thin circular ring, radii r_1, r_2	Through center perpendicular to plane of ring	$\frac{m(r_1^2+r_2^2)}{2}$
Thin circular ring, radii r_1, r_2	Through center along any diameter	$\frac{m(r_1^2+r_2^2)}{4}$
Uniform solid sphere, radius r	Any diameter	$\frac{2mr^2}{5}$
Thin spherical shell, radius r	Any diameter	$\frac{2mr^2}{3}$
Spherical shell, radii r_1, r_2	Any diameter	$\frac{2m(r_1^5-r_2^5)}{(r_1^3-r_2^3)}$
Right cone, radius of base r	Along axis of cone	$\frac{3mr^2}{10}$
Uniform circular cylinder, length l, radius r	Through center perpendicular to axis	$m\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$
Hollow circular cylinder, length l, radii r_1, r_2	Through center perpendicular to axis	$m\left(\frac{r_1^2+r_2^2}{4}+\frac{l^2}{12}\right)$
Very thin circular cylinder, length l, radius r	Through center perpendicular to axis	$m\left(\frac{r^2}{2}+\frac{l^2}{12}\right)$

6.11 A More General Analysis of the Motion of Rigid Bodies

In this section we shall show the importance of the center of mass in dynamics. We have already proved that for linear motion the center of mass of a body moves as though all the mass of the body were concentrated at that point and the resultant external force acted there. We shall now show that there is the same relationship between the rotational acceleration and the applied torques about an axis

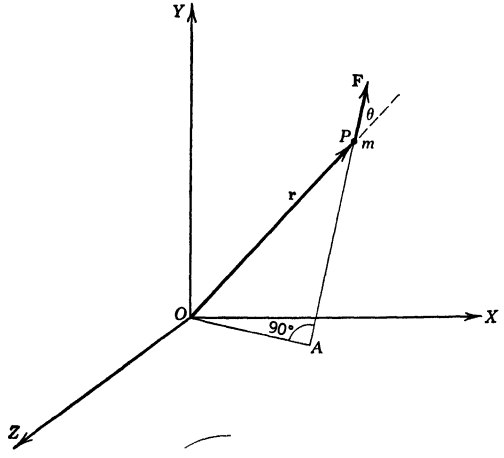


FIG. 6.15 Rotational motion of a body with reference to a fixed point.

through the center of mass as there is about a fixed axis. Finally we shall show that for a body having both translational and rotational motion the total kinetic energy may be expressed either as the sum of the translational kinetic energy of the center of mass and the rotational kinetic energy about the center of mass, or as the rotational kinetic energy about the instantaneous axis of rotation.

Let us consider a particle of mass m acted on by a force F as shown in Fig. 6.15. Relative to an origin O fixed in an inertial system, the force F exerts a torque L which has a magnitude of

$$F \times OA = Fr \sin \theta$$

and a direction that would tend to produce counterclockwise rotation about O. This torque L can be expressed as the vector or *cross product* of the vectors r and F. This product is written as

$$\mathbf{L} = \mathbf{r} \times \mathbf{F} \tag{6.16}$$

By definition the vector product of two vectors is a third vector whose magnitude is the product of the magnitudes of the two vectors and the sine of the angle between their positive directions, and whose direction is given by the direction in which a right-handed screw would progress when rotated from the first to the second vector. If the vectors all lie in the plane of this paper, then rotating a right-handed screw from the positive direction of $\mathbf{r}$ into $\mathbf{F}$, that is, in a counterclockwise direction, would cause the screw to progress perpendicularly out of the plane of the paper. Notice that the positive direction of each vector must be used when considering the rotation. Applying this definition to the vectors $\mathbf{r}$ and $\mathbf{F}$ in Fig. 6.15, we see that their vector product has a magnitude of $rF \sin \theta$ and is directed perpendicularly out of the plane of the paper corresponding to the counterclockwise rotation of $\mathbf{r}$ to $\mathbf{F}$.

Before proceeding further with the dynamical theory, let us consider some of the properties of this vector product. The vector product of a vector and itself such as $(\mathbf{r} \times \mathbf{r})$ is zero since the angle between the vectors is zero. Thus for the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along the X, Y, Z axes respectively it follows that

$$\mathbf{i} \times \mathbf{i} = 0 \qquad \mathbf{j} \times \mathbf{j} = 0 \qquad \mathbf{k} \times \mathbf{k} = 0$$

Also, by applying the rule to a system of right-handed axes X, Y, Z it can be seen that, if the unit vector $\mathbf{i}$ is rotated into the unit vector $\mathbf{j}$, there results a unit vector $\mathbf{k}$ along the Z axis. Thus

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

and

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

The results can readily be recalled by noticing that if the vectors are taken in the cyclical order $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the cross-product of two of them is the positive third unit vector, whereas if taken in a reverse direction the cross-product is negative.

By writing out the vectors $\mathbf{r}$ and $\mathbf{F}$ in component form their cross-product can be written as a determinant. Suppose that the coordinates of a point P are x, y, z relative to the respective fixed axes X, Y, Z with the origin at O. The vector $\mathbf{r}$ can be expressed

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

Also, if the components of the force $\mathbf{F}$ acting at P are F_x, F_y, F_z respectively, then

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

The vector product is

$$\begin{aligned}\mathbf{r} \times \mathbf{F} &= (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \times (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z) \\ &= \mathbf{k}xF_y - \mathbf{j}xF_z - \mathbf{k}yF_x + \mathbf{i}yF_z + \mathbf{j}zF_x - \mathbf{i}zF_y \\ &= \mathbf{i}(yF_z - zF_y) + \mathbf{j}(zF_x - xF_z) + \mathbf{k}(xF_y - yF_x)\end{aligned}$$

which may be written in the form of a determinant, thus:

$$\mathbf{L} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{6.17}$$

Let us now return to the problem of the particle of mass m in Fig. 6.15 acted on by the force $\mathbf{F}$ which produces a torque $\mathbf{r} \times \mathbf{F}$ about the fixed point O. Suppose that the instantaneous angular velocity

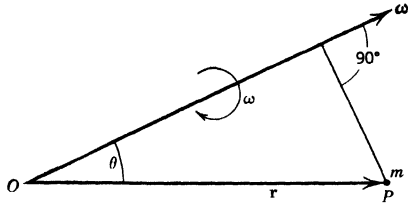


FIG. 6.16 The linear velocity of m about O due to the angular velocity ω.

of the mass m about the fixed point O is $\boldsymbol{\omega}$. Owing to this angular velocity $\boldsymbol{\omega}$ the particle of mass m has an instantaneous linear velocity $\mathbf{t}$ which from Fig. 6.16 may be seen to be

$$\mathbf{t} = \boldsymbol{\omega} \times \mathbf{r}$$

The magnitude of $\mathbf{t}$ is $\omega r \sin\theta$ and is in a direction at right angles to the plane containing the vectors $\boldsymbol{\omega}$ and $\mathbf{r}$. In this example we are considering the motion to be entirely one of rotation. Thus the vector $\mathbf{t}$ in Fig. 6.15 is perpendicular to $\mathbf{r}$ in the plane containing the vectors $\mathbf{r}$ and $\mathbf{F}$. The linear momentum of the mass m is $m\mathbf{t}$, and the angular momentum or moment of momentum $\boldsymbol{\Omega}$ of m about the axis through the fixed point O is defined as

$$\boldsymbol{\Omega} = \mathbf{r} \times m\mathbf{t} \tag{6.18}$$

We shall now show that the torque $(\mathbf{r} \times \mathbf{F})$ exerted on the mass m in Fig. 6.15 is equal to the time rate of change of angular momentum of the mass m. By Newton's second law, $\mathbf{F} = m\dot{\mathbf{v}}$. Hence the torque L about O is

$$\mathbf{L} = (\mathbf{r} \times \mathbf{F}) = (\mathbf{r} \times m\dot{\mathbf{v}})$$

Now by carrying out the differentiation in the usual manner it follows that

$$(\mathbf{r} \times m\ddot{\mathbf{r}}) = \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}})$$

since $(\dot{\mathbf{r}} \times m\dot{\mathbf{r}}) = 0$ or the cross-product of a vector and itself is zero. Hence from the above equations

$$\mathbf{L} = \frac{d\boldsymbol{\Omega}}{dt} = \dot{\boldsymbol{\Omega}} \tag{6.19}$$

or the torque about any fixed point O as origin is equal to the time rate of change of the angular momentum of the particle about O.

Let us now consider the rotation of a rigid body relative to some fixed point which is produced by a torque. Let the mass m in Fig. 6.15 be one of the particles of the body, as for instance the ith, having a mass m_i at a distance $\mathbf{r}_i$ from the fixed point O. In other words we are considering the particle m to be a typical particle m_i of the body. The particle m_i has an external force $\mathbf{F}_i$ and also an internal force acting on it. Both of these exert torques about the fixed point O. The sum of the moments of all the external forces acting on all the particles of the body is the total moment $\mathbf{L}_0$ of the applied forces, since we have assumed that the sum of all the internal forces acting on the particles composing a rigid body is zero. If the sum of the internal forces were not zero, then the body would spontaneously accelerate. In other words the sum of the moments of all the internal forces acting on the particles about the fixed point O is zero. Hence the sum of all the moments of the external forces about O is

$$\mathbf{L}_0 = \sum_1^n \mathbf{r}_i \times \mathbf{F}_i$$

Now

$$\mathbf{F}_i = m_i\ddot{\mathbf{r}}_i$$

Hence

$$\mathbf{L}_0 = \sum_1^n (\mathbf{r}_i \times m_i\ddot{\mathbf{r}}_i)$$

$$= \sum_1^n \frac{d}{dt}(\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i)$$

since

$$\dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i = 0$$

The total angular momentum $\mathbf{\Omega}_0$ of the body about the fixed point O is defined as

$$\mathbf{\Omega}_0 = \sum_1^n (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i)$$

Hence

$$\mathbf{L}_0 = \frac{d\mathbf{\Omega}_0}{dt} = \dot{\mathbf{\Omega}}_0 \tag{6.20}$$

Thus the resultant external torque acting on a body about a fixed point is equal to the time rate of increase of angular momentum of the body about the fixed point. This principle applies equally well to either a group of particles or a rigid body so long as the internal forces

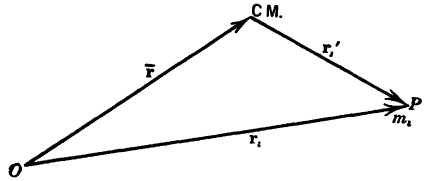


FIG. 6.17 Rotation of a body relative to its center of mass

act along the lines joining them. If there are no external torques acting on a group of particles or on a rigid body, then from Eqn. 6.20 it follows that the total angular momentum of the particles or the rigid body about any fixed point as origin is a constant quantity. Thus the angular momentum is conserved.

It will now be shown that a similar expression can be obtained for rotation about the center of mass of a body. Let P be the position of a particle of mass m_i in a body that is rotating about an axis through the point O fixed in an inertial system. In Fig. 6.17 the position of the center of mass of the body is at the point marked C.M., and the position of a particle m_i at P is given by the vector $\mathbf{r}_i$ relative to the fixed point O and by the vector $\mathbf{r}_i'$ relative to the moving center of mass of the body. If $\bar{\mathbf{r}}$ is the vector distance of the center of mass from the fixed point O, then at any instant of time

$$\mathbf{r}_i = \bar{\mathbf{r}} + \mathbf{r}_i'$$

The total angular momentum $\mathbf{\Omega}_0$ of the body about the fixed point O is by definition

$$\boldsymbol{\Omega}_0 = \sum_1^n (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i)$$

$$= \sum_1^n [(\bar{\mathbf{r}} + \mathbf{r}_i') \times m_i(\dot{\bar{\mathbf{r}}} + \dot{\mathbf{r}}_i')]$$

$$= (\bar{\mathbf{r}} \times M\dot{\bar{\mathbf{r}}}) + \dot{\bar{\mathbf{r}}} \times \left(\sum_1^n m_i \dot{\mathbf{r}}_i'\right) + \left(\sum_1^n \mathbf{r}_i'\right) \times M\dot{\bar{\mathbf{r}}} + \sum_1^n (\mathbf{r}_i' \times m_i \dot{\mathbf{r}}_i')$$

where M is the total mass $\sum_1^n m_i$ of the body and $\bar{\mathbf{r}}$ is a constant. Since $\mathbf{r}_i'$ is measured relative to the center of mass of the body, it follows that $\sum_1^n m_i \mathbf{r}_i'$ must be zero for the whole body and it also follows that

$$\sum_1^n m_i \dot{\mathbf{r}}_i' = \frac{d}{dt} \sum_1^n m_i \mathbf{r}_i' = 0$$

Thus the two middle terms in the expansion are zero and the total angular momentum $\boldsymbol{\Omega}_0$ is

$$\boldsymbol{\Omega}_0 = \bar{\mathbf{r}} \times M\dot{\bar{\mathbf{r}}} + \sum_1^n (\mathbf{r}_i' \times m_i \dot{\mathbf{r}}_i')$$

Now the total angular momentum of the body about its center of mass $\boldsymbol{\Omega}_{C.M}$ is by definition

$$\boldsymbol{\Omega}_{C.M} = \sum_1^n (\mathbf{r}_i' \times m_i \dot{\mathbf{r}}_i')$$

Hence

$$\boldsymbol{\Omega}_0 = \bar{\mathbf{r}} \times M\dot{\bar{\mathbf{r}}} + \boldsymbol{\Omega}_{C.M}. \tag{6.21}$$

Let us now consider the total torque $\mathbf{L}_0$ exerted about O which is given by Eqn. 6.16 as

$$\mathbf{L}_0 = \sum_1^n (\mathbf{r}_i \times \mathbf{F}_i)$$

$$= \bar{\mathbf{r}} \times \sum_1^n \mathbf{F}_i + \sum_1^n (\mathbf{r}_i' \times \mathbf{F}_i)$$

Now $\sum_1^n (\mathbf{r}_i' \times \mathbf{F}_i)$ is equal to the torque $\mathbf{L}_{C\,M.}$ exerted by the external forces about the center of mass of the body so that

$$\mathbf{L}_0 = \bar{\mathbf{r}} \times \sum_1^n \mathbf{F}_i + \mathbf{L}_{C\,M} \tag{6.22}$$

Since the torque $\mathbf{L}_0$ and the total angular momentum $\mathbf{\Omega}_0$ are taken with reference to the fixed origin O, we can apply Eqn. 6.20, namely $\mathbf{L}_0 = \dot{\mathbf{\Omega}}_0$. Substituting the values of $\mathbf{\Omega}_0$ and $\mathbf{L}_0$ from Eqns. 6.21 and 6.22 we have

$$\begin{aligned} \bar{\mathbf{r}} \times \sum_1^n \mathbf{F}_i + \mathbf{L}_{C\,M} &= \frac{d}{dt}(\bar{\mathbf{r}} \times M\dot{\bar{\mathbf{r}}}) + \dot{\mathbf{\Omega}}_{C.M.} \\ &= \bar{\mathbf{r}} \times M\ddot{\bar{\mathbf{r}}} + \dot{\mathbf{\Omega}}_{C\,M} \end{aligned}$$

since the vector product $(\dot{\bar{\mathbf{r}}} \times \dot{\bar{\mathbf{r}}})$ of a vector and itself is zero. From Eqn. 6.3 for the translational motion of the center of mass of the body it follows that

$$\sum_1^n \mathbf{F}_i = M\ddot{\bar{\mathbf{r}}} \qquad \text{and} \qquad \bar{\mathbf{r}} \times \sum_1^n \mathbf{F}_i = \bar{\mathbf{r}} \times M\ddot{\bar{\mathbf{r}}}$$

Hence by subtraction we have

$$\mathbf{L}_{C.M.} = \dot{\mathbf{\Omega}}_{C\,M.} \tag{6.23}$$

or *the torque about the center of mass of a body is equal to the time rate of change of the angular momentum about the center of mass.* This relationship between the torque and the rate of change of angular momentum holds whether these quantities are computed with reference to a point fixed in an inertial system or to the moving center of mass of the body.

We now have two important relationships which hold with respect to the center of mass of a body. The first of these, given in Eqn. 6.3,

$$\mathbf{F} = M\ddot{\bar{\mathbf{r}}}$$

shows that the linear or translational acceleration of the center of mass of a body is the same as though all the mass of the body were concentrated at the center of mass and the resultant external force also acted at that point. The second relationship given in Eqn. 6.23 shows that the body turns about the center of mass as though the center of mass were fixed and the same system of forces continued to act on the body as were already acting. These two important principles regarding the translational and rotational motion of the

center of mass can frequently be applied in solving problems, inasmuch as the translational and rotational motions about the center of mass of a body are independent of each other.

6.12 The Kinetic Energy of a Rotating Body

We shall now find the kinetic energy of a body rotating with respect to a fixed point and also with respect to its center of mass. Taking the fixed point O, Fig. 6.17, as the origin of the polar coordinates, the kinetic energy T of the body about this fixed point is

$$T = \sum_1^n \tfrac{1}{2} m_i \dot{r}_i^2 \tag{6.24}$$

This kinetic energy can be written in terms of the translational and rotational kinetic energies of the center of mass by using the relationship shown in Fig. 6.17, namely

$$\mathbf{r}_i = \bar{\mathbf{r}} + \mathbf{r}_i'$$

From this it follows by differentiation that

$$\dot{\mathbf{r}}_i = \dot{\bar{\mathbf{r}}} + \dot{\mathbf{r}}_i'$$

If the dot or scalar product of each side of this equation and itself is taken, then the kinetic energy T in Eqn 6.24 can be written

$$\begin{aligned} T &= \sum_1^n \tfrac{1}{2} m_i (\dot{r}_i^2) = \sum_1^n \tfrac{1}{2} m_i (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \\ &= \sum_1^n [\tfrac{1}{2} m_i (\dot{\bar{\mathbf{r}}} + \dot{\mathbf{r}}_i') \cdot (\dot{\bar{\mathbf{r}}} + \dot{\mathbf{r}}_i')] \\ &= \tfrac{1}{2} M \dot{\bar{r}}^2 + \sum_1^n \tfrac{1}{2} m_i \dot{r}_i'^2 + \dot{\bar{\mathbf{r}}} \cdot \sum_1^n m_i \dot{\mathbf{r}}_i' \end{aligned}$$

where M is the total mass $\sum_1^n m_i$ of the body and $\dot{\bar{r}}$ is a constant. Since $\mathbf{r}'$ is measured relative to the center of mass of the body, it follows that

$$\sum_1^n m_i \mathbf{r}_i' = 0 \qquad \text{and} \qquad \sum_1^n m_i \dot{\mathbf{r}}_i' = 0$$

Thus the kinetic energy of the body relative to the origin of coordinates at its center of mass is

$$T = \tfrac{1}{2}M\dot{\mathbf{r}}^2 + \sum_1^n \tfrac{1}{2}m_i\dot{\mathbf{r}}_i'^2 \tag{6.25}$$

The first term on the right-hand side of Eqn. 6.25 is the translational kinetic energy of a particle having the mass M of the body moving with the linear velocity $\dot{\mathbf{r}}$ of the center of mass, and the second is the total kinetic energy of all the individual particles taken with respect to the center of mass.

This increase in the kinetic energy is equal to the work done by the applied forces. Suppose that the ith particle of the body undergoes a displacement $d\mathbf{r}_i$ under the action of the applied force $\mathbf{F}_i$, Fig. 6.15. Then the work done by the applied force is

$$\mathbf{F}_i \cdot d\mathbf{r}_i$$

Since the applied force $\mathbf{F}_i$ produces an acceleration $\ddot{\mathbf{r}}_i$ on the particle, the work done may be written

$$\mathbf{F}_i \cdot d\mathbf{r}_i = m\ddot{\mathbf{r}}_i \cdot d\mathbf{r}_i$$

Now there is also an internal force acting on m_i, but for the body as a whole the work done by the internal forces is zero since the internal forces are in equilibrium. Hence the work done by the applied forces on the body in the displacement is $\sum_1^n \mathbf{F}_i \cdot d\mathbf{r}_i$. From the above this work can be written

$$\begin{aligned}\sum_1^n \mathbf{F}_i \cdot d\mathbf{r}_i &= \sum_1^n m_i\ddot{\mathbf{r}}_i \cdot d\mathbf{r}_i \\ &= \sum_1^n \frac{1}{2}\frac{d}{dt}(m\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)\,dt \\ &= \sum_1^n \frac{1}{2}\,d(m_i\dot{r}_i^2)\end{aligned}$$

Suppose that the body undergoes a displacement from $\mathbf{r}_1$ to $\mathbf{r}_2$ and that in consequence the velocity changes from ${}_1\mathbf{v}_i$ to ${}_2\mathbf{v}_i$. Then by integration we have for the work done

$$\sum_1^n \int_{r_1}^{r_2} \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_1^n \tfrac{1}{2}(m_i\,{}_1v_i^2 - m_i\,{}_2v_i^2) \tag{6.26}$$

Thus the total work done on the body in a given displacement is equal to the change in kinetic energy of the body during the displacement. These results are calculated with reference to the fixed point O in Fig. 6.15. In order to give a better understanding of these principles we shall apply them first to a problem in which there is a fixed axis of rotation and second to the problem in which there is an instantaneous axis of rotation which maintains a fixed direction in space. An example of the latter is a cylinder rolling down an inclined plane, Fig. 6.18.

For the first problem let us suppose that the body in Fig. 6.15 is rotating about a *fixed axis* through the point O with an angular velocity ω perpendicular to the plane of the paper. For this situation $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ has a magnitude of ωr_i, and the kinetic energy T of the body about the fixed axis from Eqn. 6.24 is

$$T = \sum_1^n \frac{1}{2} m_i \dot{r}_i^2 = \sum_1^n \frac{1}{2} m_i r_i^2 \omega^2$$

$$= \frac{1}{2}\left[\omega^2 \sum_1^n m_i r_i^2\right]$$

$$= \frac{I_0 \omega^2}{2} \tag{6.27}$$

since ω is the same for every particle in the body and $\sum_1^n m_i r_i^2$ is by definition the moment of inertia I_0 of the body about the axis through O about which the rotation is taking place.

Relative to the center of mass the kinetic energy is given by Eqn. 6.25.

$$T = \tfrac{1}{2} M \dot{\bar{r}}^2 + \sum_1^n \tfrac{1}{2} m_i \dot{r}_i'^2$$

The first term on the right is the translational kinetic energy of the center of mass, and the second is the total kinetic energy of all the particles in the body relative to the center of mass. This second term is equal to $I_{\mathrm{C.M.}}\omega^2/2$. Hence the kinetic energy of a body rotating about a fixed axis may be written

$$T = \tfrac{1}{2} I_0 \omega^2 = \tfrac{1}{2} M \dot{\bar{r}}^2 + \tfrac{1}{2} I_{\mathrm{C.M}}\, \omega^2 \tag{6.28}$$

where I_0 is the moment of inertia of the body about the fixed axis through the fixed point O, and $I_{C.M.}$ is the moment of inertia of the body about its center of mass. This kinetic energy can be equated to the work done by the applied forces as is done in Eqn. 6.26.

6.13 Example: A Cylinder Rolling Down a Rough Plane

Let us consider the motion of a uniform cylinder of mass M and radius R rolling down a rough, inclined plane whose length is l and whose angle of inclination is θ. The problem is to find the linear and the angular acceleration of the cylinder, its energy at the bottom of

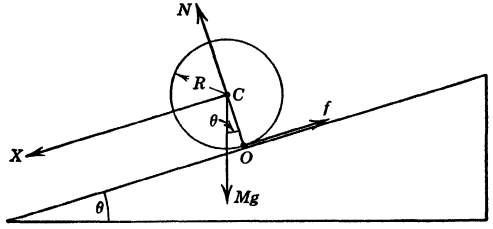


FIG. 6.18 Cylinder rolling down an inclined plane.

the plane, and the time taken for it to roll down the plane. We shall assume that on the rough plane the force of friction between the cylinder and the plane is large enough to cause the cylinder to roll without sliding. This force of friction, shown in Fig. 6.18 as f, is exerted by the plane on the cylinder and causes the line of contact of cylinder and plane through O to be instantaneously at rest. Consider what would happen if this were not the case!

If the angular acceleration of the cylinder about the axis of rotation through O is α, then the linear acceleration $\ddot{x}$ of the center of mass down the plane is

$$\ddot{x} = \alpha R$$

If the cylinder is rolling and not sliding, the angular velocity and the acceleration about the axis through O must be respectively equal to the angular velocity and acceleration about the axis through the center of mass C. In order to satisfy yourself on this point consider the displacement of C and O when the cylinder rolls a short distance down the plane.

To obtain the angular acceleration we shall use the relationship $L_0 = I_0\alpha$ for the instantaneous motion of the cylinder about the axis through O. Thus

$$MgR \sin \theta = I_0\alpha \tag{6.29}$$

The moment of inertia about a horizontal axis through C is the same as that for a circular disk. From Eqn. 6.12

$$I_{\text{C M.}} = \frac{MR^2}{2}$$

and by the theorem of parallel axes

$$I_0 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

Substituting in Eqn. 6.29 gives the angular acceleration α as

$$\alpha = \frac{2g \sin \theta}{3R}$$

and the acceleration of the center of mass parallel to the plane as

$$\ddot{x} = \alpha R = \tfrac{2}{3}g \sin \theta \tag{6.30}$$

This result may also be obtained by considering the motion of the center of mass. From Eqns. 6.3 and 6.23 for the translational and rotational accelerations of the center of mass, we have

$$M\ddot{x} = Mg \sin \theta - f$$

and

$$fR = \dot{\Omega}_{\text{C M}} = I_{\text{C M}}\alpha$$

(You should prove that in this problem $\dot{\Omega}_{\text{C M}} = I_{\text{C M}}\alpha$.) Substituting for α and $I_{\text{C.M.}}$ gives

$$I_{\text{C.M.}}\alpha = \frac{MR^2\ddot{x}}{2R} = \frac{MR\ddot{x}}{2}$$

Hence

$$f = \frac{M\ddot{x}}{2}$$

Substituting this value for f in the above equation, we have

$$M\ddot{x} = Mg \sin \theta - \frac{M\ddot{x}}{2}$$

or

$$\ddot{x} = \tfrac{2}{3}g \sin \theta$$

which agrees with Eqn. 6.30.

Consider now the kinetic energy of the cylinder as it rolls down the plane. Let us assume that the cylinder starts from rest at the top of the plane and after a time t_f is at the bottom of the plane where it has an angular velocity of ω_f and a linear velocity of the center of mass equal to v. Thus at $t = 0$, $x = 0$, $\dot{x} = 0$, and $\omega = 0$; at $t = t_f$,

$x = l$, $\dot{x} = v$, and $\omega = \omega_f$. The velocity v at the bottom of the plane can be obtained from Eqn. 6.30. Multiplying this equation by the identity $\dot{x}\,dt = dx$ gives

$$\tfrac{1}{2}d(\dot{x}^2) = \tfrac{2}{3}g \sin\theta\, dx$$

Integrating and applying the boundary conditions gives

$$\frac{v^2}{2} = \frac{2}{3}gl \sin\theta \tag{6.31}$$

Now this result can also be obtained from Eqn. 6.26, which, applied to this problem, gives for the work done and the change in kinetic energy

$$Mgl \sin\theta = \tfrac{1}{2}I_0\omega_f^2 = \tfrac{1}{2}I_{C\,M}\,\omega_f^2 + \tfrac{1}{2}Mv^2$$

where we have taken the work done by the external force $\int \mathbf{F} \cdot d\mathbf{r}$ as equal to the loss in potential energy $Mgl \sin\theta$. Substituting for I_0 and $I_{C.M.}$ and setting $\omega_f^2 = v^2/R^2$ we have

$$Mgl \sin\theta = \frac{3MR^2v^2}{4R^2}$$

or

$$gl \sin\theta = \tfrac{3}{4}v^2$$

Thus from the expression for the energy we have

$$v^2 = \tfrac{4}{3}gl \sin\theta$$

which agrees with Eqn. 6.31.

The time taken for the cylinder to roll down the plane is obtained by integrating Eqn. 6.30 with respect to time. Integrating $\ddot{x} = \frac{2}{3}g \sin\theta$ gives

$$\dot{x} = \tfrac{2}{3}tg \sin\theta + C_1$$

where C_1 is zero from the boundary conditions. Integrating a second time gives

$$\bar{x} = \frac{t^2}{3}\, g \sin\theta + C_2$$

where the boundary conditions give $C_2 = 0$ and $l = (t_f^2/3)g \sin\theta$ or

$$t_f = \sqrt{\frac{3l}{g \sin\theta}}$$

In obtaining the value of the velocity v at the bottom of the plane from the energy equation we assumed that the work done by the external force was $Mgl \sin\theta$ and that no work was done by or against

the frictional force f. The force of friction f brings the cylinder instantaneously to rest along the line of contact through O. There is no work done by this force of friction since there is no displacement of the cylinder in the direction of the force. If work were done, then the cylinder would be sliding and not rolling.

6.14 Example: Rotation of a Heavy Pulley

Suppose that a uniform pulley of mass M and radius R is free to turn about a frictionless axle through its center. Two masses M_1 and M_2 are suspended at the end of a weightless inextensible cord passing over the pulley, Fig. 6.19. The problem is to find the acceleration of this system.

If mass M_1 is larger than mass M_2, then M_1 will descend with some acceleration and M_2 will ascend with an equal acceleration. Why must this be so? The pulley will rotate only if there is friction between the cord and the periphery of the pulley. If this is assumed to be so, the tensions in the cord on the two sides of the pulley must be different. They are denoted by T_1 and T_2, and from the direction of the accelerations it follows that T_1 must be greater than T_2. The upper arrow for T_1, Fig. 6.19, represents the direction of the force exerted by the cord on the pulley, and the lower arrow the direction of the force exerted by the cord on mass M_1.

If we apply Newton's second law to each of the masses M_1 and M_2 respectively, we have for the linear acceleration a

$$M_1 g - T_1 = M_1 a$$

$$T_2 - M_2 g = M_2 a$$

Taking moments about the axis of rotation O

$$T_1 R - T_2 R = I_0 \alpha = \frac{MR^2 \alpha}{2}$$

where α is the angular acceleration of the pulley and I_0 the moment of inertia of the pulley about the axis through O. But $\alpha = a/R$. Hence

$$(T_1 - T_2) = \frac{Ma}{2}$$

Combining the upper two equations, we have

$$(M_1 - M_2)g - (T_1 - T_2) = a(M_1 + M_2)$$

or

$$(M_1 - M_2)g = a\left(M_1 + M_2 + \frac{M}{2}\right)$$

or

$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + \dfrac{M}{2}} = \frac{(M_1 - M_2)g}{[M_1 + M_2 + (I_0/R^2)]}$$

Thus the effect of the mass of the pulley is to increase the inertia or effective mass of the system by I_0/R^2 or $M/2$.

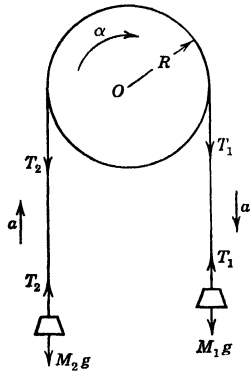


FIG. 6.19 Rotation of a heavy pulley

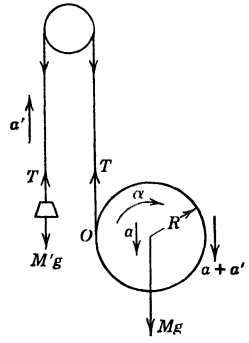


FIG. 6.20 Cord unwinding from a cylinder.

6.15 Example: A Cord Unwinding from a Cylinder, as in a Yo-Yo

A cord is wound around a cylinder and the free end of the cord is passed over a fixed smooth peg with a weight M' attached to this end, Fig. 6.20. Suppose that the cylinder has a mass M and a radius R. The cylinder has an instantaneous axis of rotation about O, the point of contact of the cord and cylinder. Let the constant angular acceleration about O be α. Then the axis of the cylinder has a linear acceleration $a = \alpha R$ relative to O. Suppose that mass M' moves upward with an acceleration a'. Then a' is the acceleration of M' relative to our assumed inertial system of which the peg is a part. Since the point O moves downward with this acceleration a', the acceleration of the center of mass of the cylinder relative to the peg, sometimes called the absolute acceleration, is $(a + a')$. The problem is to find a, a', and also T, the tension in the cord, in terms of M, M', and R.

Consider first the angular acceleration and the torque about the center of mass of the cylinder. From Fig 6 20 we see that

$$TR = I_{\text{C.M.}}\alpha = I_{\text{C.M.}}\frac{a}{R}$$

For a uniform cylinder $I_{\text{C M}} = MR^2/2$ so that

$$T = \frac{Ma}{2}$$

From Newton's second law the linear acceleration of the center of mass of the cylinder is

$$Mg - T = M(a + a')$$

Similarly for the mass M'

$$T - M'g = M'a'$$

These three equations are the dynamical equations of motion of the system. From them we obtain our required quantities. Substituting for T gives

$$g = \frac{3a}{2} + a'$$

and

$$\frac{Ma}{2} = M'a' + M'g \qquad \text{or} \qquad a' + g = \frac{M}{2M'}a$$

Thus

$$a = \frac{4gM'}{(M + 3M')} \qquad a' = \frac{g(M - 3M')}{(M + 3M')} \qquad T = \frac{2gMM'}{(M + 3M')}$$

We could also have taken moments about O, giving

$$M(g - a')R = I_0\alpha = \frac{3R^2Ma}{2R}$$

Notice that the moment of the force about O is not MgR but $M(g - a')R$. Why? Thus

$$(g - a') = \tfrac{3}{2}a$$

which can be obtained by adding the equations for the angular and linear accelerations of the center of mass. From these equations it follows that, if $3M'$ is larger than M, the acceleration of the hanging weight will be downward.

6.16 The Stability of a Car on a Horizontal Turn

Suppose that a car of mass M having a speed v is making a horizontal turn to the left, Fig. 6.21. The radius of this turn is R. The force

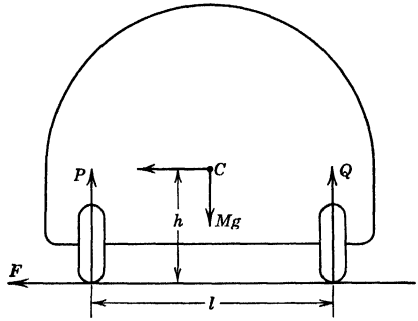


FIG. 6.21 Forces on a car making a horizontal turn.

providing the centripetal acceleration of Mv^2/R is the force of friction F between the wheels and the ground. Thus

$$F = \frac{Mv^2}{R}$$

Since there is no vertical acceleration

$$P + Q = Mg$$

Also, since the angular acceleration α about the center of mass at C is zero, it follows that the torque about C is zero.

$$\frac{Pl}{2} - \frac{Ql}{2} + Fh = 0$$

Hence

$$Q - P = \frac{2hMv^2}{lR}$$

and by addition of these equations

$$2P = Mg - \frac{2hMv^2}{lR}$$

Now P is negative or there is no support on the inside wheel if

$$\frac{2hv^2}{lR} > g$$

or

$$v^2 > \frac{lRg}{2h}$$

Thus, if v is large enough, the wheel on the inside of the curve leaves the ground and the car tends to overturn outward.

The force of friction F between the tires and the road can be eliminated by banking the road as shown in Fig. 6.22. It is a simple

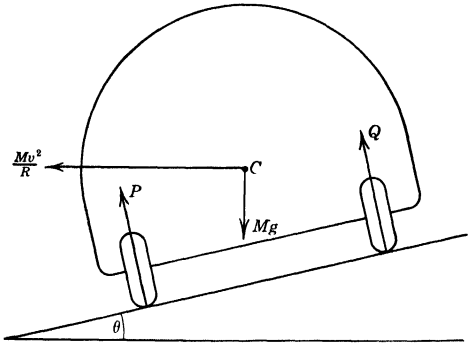


FIG 6.22 Car making a turn on a banked road

problem to find this appropriate angle of banking of the road θ for a car turning a corner of radius R with a speed v. In this example the centripetal force necessary for the turn is provided by the horizontal components of the forces P and Q acting on the wheels

$$\frac{Mv^2}{R} = (P + Q) \sin \theta$$

The vertical components must equal the weight

$$Mg = (P + Q) \cos \theta$$

Thus, if there is to be no horizontal force of friction between the road and the wheels, the road must be banked at an angle θ given by

$$\tan \theta = \frac{v^2}{Rg}$$

6.17 The Physical or Compound Pendulum

In Chapter 5 we discussed the ideal simple pendulum which was assumed to consist of a point mass suspended by a weightless cord

from a rigid support. We shall now drop some of these restrictions and investigate the motion of a rigid body oscillating about a fixed frictionless support. The rigid body might be a uniform rod such as a meter stick oscillating in a vertical plane about a fixed horizontal axis through a point near one end as in Fig. 6.23.

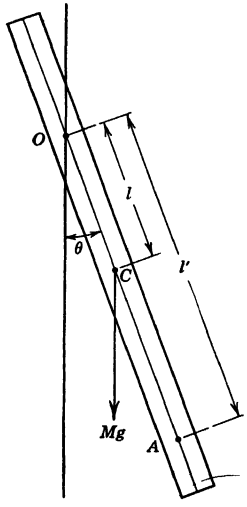


FIG. 6.23 A physical pendulum showing the center of suspension O and the center of oscillation A.

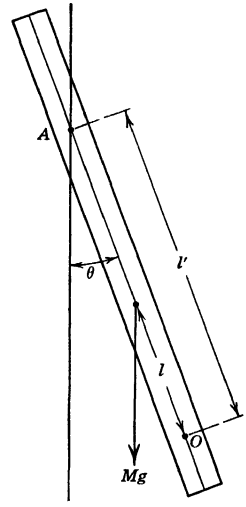


FIG. 6.24 The same physical pendulum with the positions of the center of suspension and oscillation reversed.

Let the rod have a mass M and be suspended at O from a frictionless support a distance l from the center of mass C. Suppose that at some instant of time the rod makes an angle θ with the vertical. At this time the torque L_0 exerted by the rod about the axis through O is

$$L_0 = Mgl \sin \theta$$

This torque is related to the angular acceleration $\ddot{\theta}$ by Eqn. 6.9, which in this case gives

$$Mgl \sin \theta = -I_0\ddot{\theta} \tag{6.32}$$

where I_0 is the moment of inertia of the rod about O. The negative sign is used since the torque tends to decrease θ whereas $\ddot{\theta}$ is positive in the direction of increasing θ. We are assuming that there are no

torques due to friction or air resistance, which is, of course, only approximately correct in practice.

We shall further assume that the maximum angle of oscillation is so small that sin θ may be replaced by the angle θ. If this is not done, then the expression for the period becomes relatively complicated but may be evaluated by the same method as for the simple pendulum. With this assumption Eqn. 6.32 becomes

$$\ddot{\theta} = -\frac{Mgl\theta}{I_0}$$

This is the expression for angular simple harmonic motion, the period of which is

$$P_0 = 2\pi\sqrt{\frac{I_0}{Mgl}} \tag{6.33}$$

It is well to check equations such as these for dimensions, that is, to show that $\sqrt{I_0/Mgl}$ has the dimensions of time. The moment of inertia of the rod about O may be written by the theorem of parallel axes in terms of the moment of inertia about the center of mass as

$$I_0 = I_{C\,M.} + Ml^2$$

If k is the radius of gyration of the rod about the center of mass, then

$$I_{C\,M} = Mk^2$$

and

$$I_0 = M(k^2 + l^2)$$

Thus the period about O from Eqn. 6.33 is

$$P_0 = 2\pi\sqrt{\frac{k^2 + l^2}{lg}} \tag{6.34}$$

An ideal simple pendulum having this same period P_0 would have a length l' given by

$$P_0 = 2\pi\sqrt{\frac{l'}{g}}$$

This length l' is related to the length l between the point of support O and the center of mass C of the physical pendulum by

$$l' = \frac{k^2 + l^2}{l} = \frac{k^2}{l} + l \tag{6.35}$$

We shall see that this length l', called the length of the equivalent simple pendulum, has physical importance. Suppose that the pendu-

lum is now suspended from an axis through A a distance l' from O, the former axis of suspension. It is then readily shown that the period of oscillation about A is the same as that about O. The moment of inertia I_A of the rod about A in Fig. 6.24 is

$$I_A = Mk^2 + M(l' - l)^2$$

which by Eqn. 6.35 becomes

$$I_A = Mk^2 + M\left(\frac{k^2}{l}\right)^2$$

$$= \frac{Mk^2}{l^2}(l^2 + k^2)$$

The period P_A of the oscillations about A is given from Eqn. 6.33 as

$$P_A = 2\pi\sqrt{\frac{I_A}{Mg(l' - l)}} = 2\pi\sqrt{\frac{Mk^2(l^2 + k^2)l}{l^2Mgk^2}} = 2\pi\sqrt{\frac{(k^2 + l^2)}{lg}} = P_0$$

The two points O and A are called the *center of suspension* and the *center of oscillation* respectively. It follows that, if either point is the center of oscillation, the other is the center of suspension.

We shall next prove that there are two points of suspension at different distances from the center of mass and on the same side of the center of mass at which the periods are equal. From Eqn. 6.35 the length l' of a simple pendulum of a given period is

$$l' = \frac{k^2}{l} + l$$

or

$$l^2 - ll' + k^2 = 0$$

The solution of this quadratic equation is

$$l = \frac{l' \pm \sqrt{l'^2 - 4k^2}}{2}$$

so that the two lengths l_1, l_2 having the same period as that of the simple pendulum of length l' are

$$l_1 = \frac{l' + \sqrt{l'^2 - 4k^2}}{2} \qquad l_2 = \frac{l' - \sqrt{l'^2 - 4k^2}}{2} \tag{6.36}$$

The period of the compound pendulum of length l_1 is

$$P = 2\pi\sqrt{\frac{M(k^2 + l_1^2)}{Mgl_1}}$$

which may readily be shown by substitution to be equal to

$$P = 2\pi \sqrt{\frac{l'}{g}}$$

As shown in Fig. 6.25 the period of a physical pendulum varies with the distance of its center of suspension from the center of mass. When the pendulum is suspended at its center of mass, the period is infinite. The figure shows that the same period is obtained when any of the four points A, B, A', B' is the point of suspension. The distance of

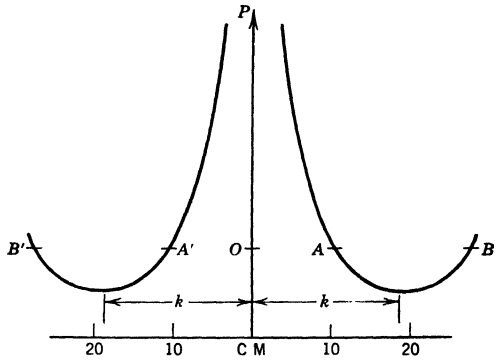


FIG. 6.25 The period P of a physical pendulum plotted against the distance from its center of mass

the point of the suspension from the center of mass for which the period is a minimum may be calculated by setting dP/dl equal to zero in the expression

$$P = 2\pi \sqrt{\frac{k^2 + l^2}{lg}}$$

When the differentiation is carried out, it is found that $dP/dl = 0$ when $l = k$. The minimum period of oscillation is therefore

$$P_{\min} = 2\pi \sqrt{\frac{2k}{g}}$$

If this minimum period is experimentally determined, then the radius of gyration k of the physical pendulum can be found. This radius of gyration k may also be found from the product of any of the two

distances l_1, l_2, for from Eqn. 6.36 it follows that

$$l_1 l_2 = k^2 \tag{6.37}$$

About 1818 an English sea captain, Kater, constructed a special form of physical pendulum to make a precise determination of the acceleration due to gravity g. In this apparatus, Fig. 6.26, there are two fixed knife edges on opposite sides of the center of mass of the pendulum. It is the object of the experiment to arrange that the periods of oscillation about each of these two knife edges are almost equal. This is done by adjusting the position of two masses, one relatively large for coarse adjustment and the other relatively small for fine adjustment.

FIG. 6.26 Kater's pendulum; Courtesy of the Central Scientific Company, Chicago.

Let us assume that the period about one knife edge at a distance of L_1 from the center of mass is P_1, and that the period about the other at a distance of L_2 is P_2. The adjustment is made so that P_1 is nearly equal to P_2. From Eqn. 6.34 the periods P_1 and P_2 are given by

$$P = 2\pi \sqrt{\frac{k^2 + L^2}{Lg}} \tag{6.34}$$

or for the length L_1

$$P_1^2 L_1 g = 4\pi^2 (k^2 + L_1^2)$$

and for the length L_2

$$P_2^2 L_2 g = 4\pi^2 (k^2 + L_2^2)$$

By subtraction

$$g(P_1^2 L_1 - P_2^2 L_2) = 4\pi^2 (L_1^2 - L_2^2)$$

or

$$\frac{4\pi^2}{g} = \frac{P_1^2 L_1 - P_2^2 L_2}{(L_1^2 - L_2^2)} = \frac{1}{2}\left[\frac{P_1^2 + P_2^2}{L_1 + L_2} + \frac{P_1^2 - P_2^2}{L_1 - L_2}\right] \tag{6.38}$$

If considerable time and care are taken, then P_1 may be made so nearly equal to P_2 that the second term is negligible. Assuming that P_1 may be taken as equal to P_2, which we shall set equal to P, then

$$\frac{4\pi^2}{g} = \frac{P^2}{L_1 + L_2}$$

or

$$g = \frac{4\pi^2}{P^2}(L_1 + L_2)$$

The distance $(L_1 + L_2)$, the distance between the knife edges, can be measured very accurately with a comparator and a standard metal meter scale. If the period is accurately known, then a really accurate determination of the value of g can be made. If the period P_1 is not exactly equal to P_2, then the first term on the right in Eqn. 6.38 can be evaluated accurately. The second term, which is very small, need not be evaluated so accurately. By balancing the pendulum on a knife edge at its center of mass, the distances L_1 and L_2 can be measured with sufficient accuracy.

Though hardly in keeping with the material in this book we shall digress to give the method by which the period of oscillation of a Kater's pendulum is determined. In science, measurements are often made by comparing a standard quantity with an unknown quantity, using a suitable instrument to make the comparison. A mass is compared with a standard mass by means of a balance; a voltage is compared with a standard voltage or electromotive force by means of a potentiometer. In this experiment the standard timing device may be an auxiliary pendulum clock whose period, of perhaps 2 sec, is checked with the time signals from the Naval Station at Arlington. The standard pendulum and Kater's pendulum are then connected in an electrical circuit so that when both are in their central positions the electrical circuit is closed and a bell rings or a lamp lights. This is known as the method of coincidences.

Suppose that after a coincidence the standard pendulum makes n half-vibrations and the Kater's pendulum makes $(n - 1)$ half-vibrations before the next coincidence. The period P' of the Kater's pendulum is larger than the period P of the standard since

$$\frac{nP}{2} = \frac{(n-1)P'}{2}$$

Half-periods are used since a coincidence takes place when the pendulums are in their central positions. Suppose that the period P of the standard pendulum is precisely 2 sec; then

$$\frac{P'}{2} = \frac{n}{n-1} = \frac{1}{\left(1 - \frac{1}{n}\right)} = \left(1 - \frac{1}{n}\right)^{-1} = 1 + \frac{1}{n} + \frac{1}{n^2} \cdots$$

If, for example, there are 552 half-vibrations of the standard pendulum

and 551 half-vibrations of the Kater's pendulum between successive coincidences, then $n = 552$ and

$$\frac{P'}{2} = \frac{552}{551} = 1 + \frac{1}{551}$$
$$= 1.001819$$

or

$$P' = 2.003638 \text{ sec}$$

6.18 Reactions on the Axis of Rotation

When a body is rotating about a fixed axis, there is present a force exerted by the rotating body on the axis. By Newton's third law there is a reacting force exerted by the axis on the body. The simplest but most common problems of this type are those in which both the forces and the body are symmetrical with respect to a plane through the center of mass perpendicular to the fixed axis. We shall assume here that the fixed axis is perpendicular to the plane of the paper and that the only external force acting on the system is that due to gravity, as shown in Fig. 6.27.

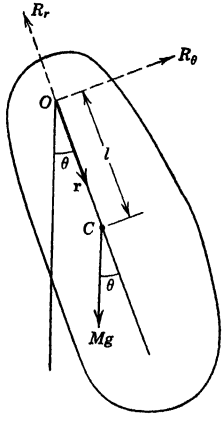


FIG. 6.27 Reactions on the axis of rotation of a physical pendulum.

Let O be the fixed axis on which the body of mass M rotates. From the assumed symmetry of the body the reactions of the axis reduce to a single force in the plane of the paper whose components, along and perpendicular to the line OC joining the axis to the center of mass, are R_r and R_θ respectively. As the body vibrates about the axis O, the line OC changes direction and so do the forces R_r and R_θ. Let O be the origin of a radius vector $\mathbf{r}$ along OC. Then from Eqns. 1.13 and 1.14 the acceleration of a particle along $\mathbf{r}$ is $(\ddot{r} - r\dot{\theta}^2)$ and perpendicular to $\mathbf{r}$ in a direction of increasing θ is $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$. For the center of mass the vector $\mathbf{r}$ has a constant magnitude of l so that $\dot{r}$ and $\ddot{r}$ are zero. From Eqn. 6.3 for the linear motion of the center of mass, we have for the equations of motion along and perpendicular to OC respectively, at any instant of time,

$$Mg \cos \theta - R_r = -Ml\dot{\theta}^2 \tag{6.39}$$

and

$$R_\theta - Mg \sin \theta = Ml\ddot{\theta} \tag{6.40}$$

where θ is the instantaneous angle made by the line OC with the vertical. The final equation of motion comes from the relationship of the torque to the angular acceleration about the fixed axis. Applying Eqn. 6.9 to this problem, we have

$$Mgl \sin \theta = -I_0 \ddot{\theta} \tag{6.41}$$

where the negative sign is used since the torque is in the direction tending to decrease θ.

The moment of inertia I_0 about O may be written in terms of the radius of gyration k about the center of mass as

$$I_0 = I_{\text{C.M.}} + Ml^2 = M(k^2 + l^2)$$

Thus Eqn. 6.41 becomes

$$\ddot{\theta} = -\frac{gl}{k^2 + l^2} \sin \theta \tag{6.42}$$

To obtain the angular velocity $\dot{\theta}$, Eqn. 6.42 is multiplied by the identity

$$d\theta = \frac{d\theta}{dt} dt = \dot{\theta}\, dt$$

giving

$$\frac{1}{2} d(\dot{\theta}^2) = -\frac{gl}{k^2 + l^2} \sin \theta \, d\theta = +\frac{gl}{k^2 + l^2} d(\cos \theta)$$

By integration

$$\frac{1}{2} \dot{\theta}^2 = \frac{gl}{k^2 + l^2} \cos \theta + C_1 \tag{6.43}$$

where C_1 is the constant of integration determined from the initial conditions of motion. In order to complete the problem let us assume that, at time $t = 0$, $\dot{\theta} = 0$ and $\theta = \pi/2$; that is, the body is started from rest with the line OC horizontal. The constant of integration C_1 is thus zero. Substituting for θ and $\ddot{\theta}$ in Eqns. 6.39 and 6.40 gives

$$R_r = Mg \cos \theta \left(1 + \frac{2l^2}{k^2 + l^2}\right)$$

and

$$R_\theta = Mg \sin \theta \left(1 - \frac{l^2}{k^2 + l^2}\right)$$

The magnitude of the resultant reaction R is

$$R = \sqrt{R_r^2 + R_\theta^2}$$

As an example of the reactions on an axis of rotation let us consider the following problem. A uniform meter stick whose mass is 150 gm is free to turn on a horizontal axis 10 cm from one of its ends. It is started from rest when the rod is at an angle of 45° above the horizontal, Fig. 6.28. The problem is to find the resultant reaction of the axis of rotation.

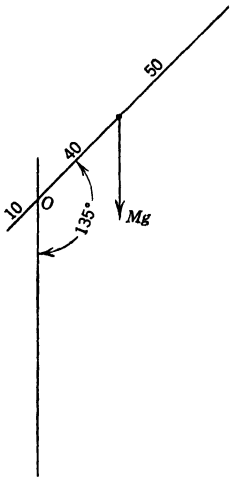


FIG. 6.28 A meter stick falling from an angle of 135° with the vertical

We must first find the radius of gyration k. For a uniform rod of length L

$$I_{G.M} = Mk^2 = \frac{ML^2}{12}$$

or

$$k^2 = \frac{L^2}{12} = \frac{10^4}{12} \text{ cm}^2$$

The constant of integration C_1 in Eqn. 6.43 is evaluated from the initial conditions: $t = 0$, $\theta = 0$, $\theta = 135°$, so that

$$C_1 = -\frac{gl}{k^2 + l^2} \cos 135° = 0.707 \frac{gl}{k^2 + l^2}$$

From Eqn. 6.43

$$\dot{\theta}^2 = \frac{2gl}{k^2 + l^2} (\cos \theta + 0.707)$$

Hence from Eqns. 6.39 and 6.40 the components of the force of reaction when the meter stick makes an angle θ with the vertical are

$$R_r = Mg \cos \theta + \frac{2Mgl^2}{k^2 + l^2} (\cos \theta + 0.707) = (347.5 \cos \theta + 139.6) \text{ gmw}$$

$$R_\theta = Mg \sin \theta \left(1 - \frac{l^2}{k^2 + l^2}\right) = 51.3 \sin \theta \text{ gmw}$$

When the meter stick is in the vertical position, $\theta = 0°$. Then R_θ is zero, and $R_r = 487.1$ gmw; that is, the reaction by the support is entirely vertical.

This value of R_r when the rod is vertical can be obtained in a simpler manner, for in this position the support must exert a force equal to the sum of the weight and the centripetal force, or

$$R_r = Mg + Ml\dot{\theta}_0^2$$

where $\dot{\theta}_0$ is the angular velocity of the rod when it is vertical. The value of $\dot{\theta}_0$ can be obtained from the conservation of energy equation. If h is the height through which the center of mass falls, then

$$Mgh = \tfrac{1}{2} I_0 \dot{\theta}_0^2$$

From Fig. 6.28

$$h = 40 \times 0.707 + 40 = 68.28 \text{ cm}$$

and

$$I_0 = M(k^2 + l^2) = M\left(\frac{29{,}200}{12}\right) \text{ gm cm}^2$$

Hence

$$\dot{\theta}_0^2 = \frac{(2 \times 68) \cdot (28 \times 12)}{29{,}200} \text{ sec}^{-2}$$

giving

$$R_r = Mg\left[1 + \frac{(2 \times 68) \cdot (28 \times 12 \times 40)}{29{,}200}\right] = 487.1 \text{ gmw}$$

6.19 Impulsive Forces

In Chapter 2 we showed that the change in momentum of a particle is equal to the time integral of the force acting on the particle or is equal to the impulse of the force. A similar theorem holds for the linear motion of the center of mass of a rigid body.

From Eqn. 6.3 the linear acceleration of the center of mass $\ddot{\mathbf{r}}$ of a body of mass M acted on by a resultant force $\mathbf{F}$ is given by

$$M\ddot{\mathbf{r}} = \mathbf{F} \tag{6.3}$$

Suppose that the force $\mathbf{F}$ is very large and that the time ΔT for which it acts is very small. Then the impulse $\mathbf{H}$ of this force is defined as

$$\mathbf{H} = \int_0^{\Delta T} \mathbf{F}\, dt$$

During the time ΔT the force may vary considerably in magnitude as when a bat strikes a ball. From Eqn. 6.3 it follows that

$$M\, d(\dot{\mathbf{r}}) = \mathbf{F}\, dt$$

If $\bar{\mathbf{v}}_0$ is the initial velocity of the center of mass when the force begins to act and $\bar{\mathbf{v}}$ that after the impulse $\mathbf{H}$ is given, then

$$\int_{\bar{\mathbf{v}}_0}^{\bar{\mathbf{v}}} M\, d(\dot{\mathbf{r}}) = \int_0^{\Delta T} \mathbf{F}\, dt = \mathbf{H}$$

or

$$M(\bar{\mathbf{v}} - \bar{\mathbf{v}}_0) = \int_0^{\Delta T} \mathbf{F}\, dt = \mathbf{H} \tag{6.44}$$

This is an equation between vector quantities. It states that *the change in momentum of the center of mass of a body in any direction is equal to the component of the impulse in that direction.*

Consider now an impulse on a body free to rotate about a fixed axis through a point O. Then from Eqn. 6.9 a torque $\mathbf{L}_0$ produces an angular acceleration $\ddot{\boldsymbol{\theta}}$ about the axis given by

$$\mathbf{L}_0 = I_0\ddot{\boldsymbol{\theta}}$$

where I_0 is the moment of inertia of the body about the axis through O. Suppose that a large torque acts for a short time. Then $\int \mathbf{L}_0\, dt$ is

called the impulse of the torque. From Eqn. 6.9 it follows that

$$\mathbf{L}_0\,dt = d(I_0\dot{\boldsymbol{\theta}}) = I_0\,d(\dot{\boldsymbol{\theta}})$$

If ω_0 is the angular velocity of the body about O just before the torque starts to act and ω that immediately after the torque has ceased acting, then

$$\int_0^{\Delta T} \mathbf{L}_0\,dt = I_0(\omega - \omega_0) \quad (6.45)$$

or *the impulse of a torque about a fixed axis is equal to the change in angular momentum about that axis.*

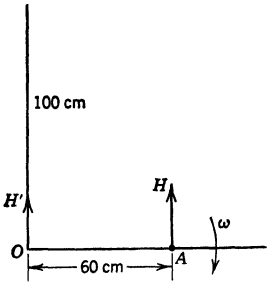


FIG. 6.29 A meter stick making an inelastic impact with rod A.

As an example of this analysis let us consider the following problem. A meter stick whose mass is 150 gm can rotate about an axis through O at one end, Fig. 6 29. Suppose that the stick starts from a vertical position and falls to the horizontal position where it strikes a fixed inelastic rod A at a distance of 60 cm from the end O. The problem is to find the impulse of the blow on the fixed inelastic rod and the impulse of the reaction at the support O.

The angular velocity ω of the rod when it is horizontal and just before it strikes the fixed rod at A is given by the law of conservation of energy

$$\frac{1}{2}I_0\omega^2 = Mg\frac{l}{2}$$

where

$$I_0 = I_{C.M} + \frac{Ml^2}{4} = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

Hence

$$\omega^2 = \frac{3g}{l} = 29.4\ \text{sec}^{-2}$$

If H is the impulse of the blow exerted on the meter stick when it strikes the rod A, then from Eqn. 6.45 for the motion about the fixed axis

$$H \times 60 = I_0\omega$$

since the meter stick comes to rest after the inelastic impact with the rod. Thus

$$H = \frac{Ml^2\omega}{180} = \frac{15 \times 10^5\sqrt{29.4}}{180} = 4.52 \times 10^4\,\frac{\text{gm cm}}{\text{sec}}$$

If H' is the reaction of the blow on the rod at the support A, then from Eqn. 6.44 for the motion of the center of mass

$$H + H' = \frac{Ml\omega}{2}$$

Thus

$$H' = (4.07 - 4.52) \times 10^4 = -4.50 \times 10^3 \frac{\text{gm cm}}{\text{sec}}$$

The blow exerted by the axis is therefore downward in the opposite direction to that shown in Fig. 6.29.

6.20 Center of Percussion

If a body that is capable of rotating freely about a fixed axis is given a blow at such a point that there is no impulsive blow on the axis, then the point is called the center of percussion of the body.

Let us consider that a uniform rod of mass M, suspended freely about an axis near one end O, is struck by a horizontal blow of impulse $\mathbf{H}$ at a distance $(x + a)$ from O as shown in Fig. 6.30. In general the blow H causes an impulsive action $-H'$ on the axis and a reaction H' on the rod. In this example H' is horizontal since H is horizontal. The problem is to find some distance x where H' is zero.

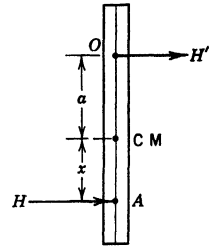


FIG. 6.30 The center of percussion of a rod.

Let ω be the angular velocity of the center of mass immediately after the blow. Then from Eqn. 6.44

$$H + H' = Ma\omega$$

From Eqn. 6.45 for the moment of the impulse about O we have

$$H(x + a) = I_0\omega$$

where

$$I_0 = I_{\text{C.M.}} + Ma^2 = Mk^2 + Ma^2 = M(k^2 + a^2)$$

Hence

$$H = \frac{M(k^2 + a^2)\omega}{x + a}$$

and

$$H' = Ma\omega - \frac{M(k^2 + a^2)\omega}{x + a} \tag{6.46}$$

If the blow is struck at the center of percussion, then by definition of the center of percussion $\mathbf{H}'$ is equal to zero. Thus the center of percussion lies at some distance x below the center of mass where

$$a - \frac{k^2 + a^2}{x + a} = 0$$

or

$$xa = k^2 \tag{6.47}$$

This is similar to equation 6.37 derived earlier for a compound pendulum. Thus, if A is the center of percussion where the impulsive blow H' on the rod is zero, then the distance OA is the length of the equivalent simple pendulum when the meter stick is regarded as a compound pendulum. In other words the center of percussion with respect to a fixed axis coincides with the center of oscillation with respect to the same fixed axis. From Eqn. 6.47 it follows that the blow H' at the axis is positive in the direction shown in Fig. 6.30 if the blow H is struck below the center of percussion A. Also, H' is negative or in the opposite direction to that shown in the figure if the blow H is struck above the center of percussion.

A rough example of the above theory, and one with which most of us are familiar, is that of a bat striking a baseball. Although the bat is held by the hands and does not move about any well-defined axis, you probably realize that there is a certain place where the bat must be held if no "sting" or impulsive blow is to be received by your hands.

6.21 An Approximate Theory of the Gyroscope

A gyroscope is any symmetrical body rotating on an axis, which axis is also free to turn about some point. A spinning top and a heavy wheel rotating on an axle supported at some point as in Fig. 6.31 are examples of gyroscopes. In these gyroscopes the axis of symmetry, about which the body rotates, precesses or rotates about another axis. Thus in Fig. 6.31*a* the wheel and axle precess with an angular velocity ω' about a vertical axis. We shall assume that the angular velocity of spin ω is very much larger than the angular velocity of precession ω' so that the total angular momentum of the system lies along the spin axis. Our approximation then consists in neglecting the angular momentum about the axis of precession.

If the wheel were not rotating it would fall vertically under gravity. When it is rotating rapidly the torque due to the weight of the wheel Mg causes precession about the vertical axis through O. The torque about O is Mgl and is represented by the vector L in Fig. 6.31*b*. This torque L is in a horizontal plane perpendicular to the plane formed by the axle and the direction of the weight Mg. If I is the moment of inertia of the wheel about the axle and ω is the angular velocity of the wheel, then the angular momentum of the wheel is $I\omega$ along the direction of the vector ω. In a short time dt the torque L produces a change in angular momentum given by Eqn. 6.45.

$$L\,dt = d(I\omega)$$

and is in a direction parallel to the torque vector L. This change in angular momentum is represented by the vector $d(I\omega)$ at right angles to $I\omega$ in Fig. 6.31*b*. The resultant of these two vectors is a vector which makes an angle $d\phi$ with the vector $I\omega$. In the time dt the

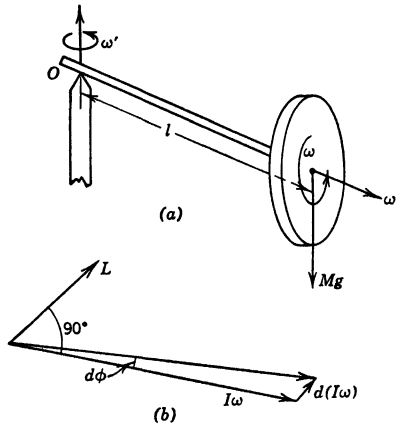


FIG. 6.31 A simple gyroscope and the components of angular momentum $I\omega$ and torque L.

vector $I\omega$ and hence the axle turns through an angle $d\phi$. This is the motion of precession. From the vector triangle in Fig. 6.31*b* it follows that

$$d(I\omega) = I\omega\, d\phi$$

Thus

$$Mgl\, dt = d(I\omega) = I\omega\, d\phi$$

or the angular velocity of precession ω' is given by

$$\omega' = \frac{d\phi}{dt} = \frac{Mgl}{I\omega} \tag{6.48}$$

From the discussion above it follows that the direction of the precession is such that the spin vector precesses towards the torque vector. If such a gyroscope as shown above is watched carefully, it will be noticed that the axle does not precess exactly in a plane but rises above and falls below this plane. This motion, called nutation, cannot be analyzed by this simple theory, but a more complete theory is given in section 6.28.

6.22 Velocity and Acceleration Relative to Moving Axes

In the discussion of Newton's laws of motion it was stated that these laws should be referred to a primary inertial system of axes such as those attached to the fixed stars. Since the earth is rotating on its axis and revolving about the sun, it follows that any system of axes attached to the earth cannot theoretically constitute an inertial system.

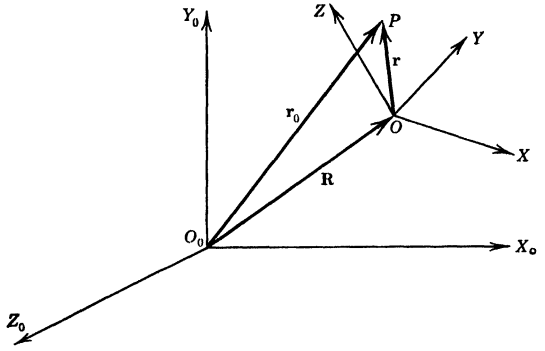


FIG. 6.32 A particle at P referred to fixed and moving axes.

We shall now investigate what changes should be made in Newton's laws when they are referred to a system of moving axes such as ones attached to the earth. This investigation will be important in rigorously analyzing the motions of a spinning top or of a gyroscope.

Suppose that O_0 is the origin of a right-handed system of axes X_0, Y_0, Z_0 which is rigidly attached to a primary inertial system. Further let O be the origin of a system of axes X, Y, Z which is moving with respect to X_0, Y_0, Z_0, Fig. 6.32. The motion of the axes X, Y, Z may involve both that of translation and that of rotation with respect to the fixed axes X_0, Y_0, Z_0.

Suppose that a point P which is moving with respect to both sets of axes has an instantaneous position denoted by the vector $\mathbf{r}_0$ with respect to the fixed axes and by the vector $\mathbf{r}$ with respect to the moving axes. If $\mathbf{R}$ is the vector distance between the origins O_0O at the instant of time being considered, then from Fig. 6.32 it may be seen that

$$\mathbf{r}_0 = \mathbf{R} + \mathbf{r}$$

Suppose that the coordinates of P with respect to the fixed set of axes are x_0, y_0, z_0. Then

$$\mathbf{r}_0 = \mathbf{i}_0 x_0 + \mathbf{j}_0 y_0 + \mathbf{k}_0 z_0 \tag{6.49}$$

where $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ are the unit vectors along the X_0, Y_0, Z_0 axes respectively. These unit vectors are constant both in magnitude and direction. Similarly, if the coordinates of P with respect to the moving set of axes are x, y, z respectively, then

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{6.50}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the X, Y, Z axes respectively. These unit vectors have a constant magnitude, namely unity, but have the varying direction of the rotating axes.

The velocity of the point P relative to the fixed axes is $\mathbf{t}_0$, and this is given by

$$\mathbf{t}_0 = \dot{\mathbf{R}} + \mathbf{t}$$

The velocity $\mathbf{t}$ is

$$\mathbf{t} = \frac{d}{dt}(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$

$$= \mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt} \tag{6.51}$$

In this expression

$$\mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z} = \mathbf{v} \tag{6.52}$$

is the apparent velocity of the point P relative to the moving axes X, Y, Z. Thus, if the point P is at rest in the moving system, then its velocity $\mathbf{v}$, relative to the axes X, Y, Z would be zero. However, even in this case the point P has a velocity relative to the fixed axes X_0, Y_0, Z_0 due to the rotation of the axes X, Y, Z. This is given by the last three terms in Eqn. 6.51.

In order to understand these terms we must investigate the meaning of such terms as $d\mathbf{i}/dt$, the time rate of change of the unit vector along the X axis of the moving axes. To do this we shall resort to geometric analysis, Fig. 6.33. In this figure let **OA** be unit vector $\mathbf{i}$ at time t and **OB** the same unit vector at time $(t + dt)$, where dt is an infinitesimal change in the time. The lines **OA** and **OB** both have a length or magnitude of unity. Since

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$

then

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

is the infinitesimal change in the unit vector $\mathbf{i}$ in the time dt, or

$$\mathbf{AB} = d\mathbf{i}$$

To complete the description of Fig. 6.33 a perpendicular BC is

drawn from B onto the XZ plane and CA is drawn in the XZ plane. Thus

$$\mathbf{AB} = \mathbf{AC} + \mathbf{CB} \tag{6.53}$$

Now **AC** represents a rotation about the $-Y$ axis in the time dt and is parallel to the Z axis. Hence

$$\mathbf{AC} = -\omega_y \, dt\mathbf{k}$$

where ω_y is the angular velocity of rotation about the Y axis. Simi-

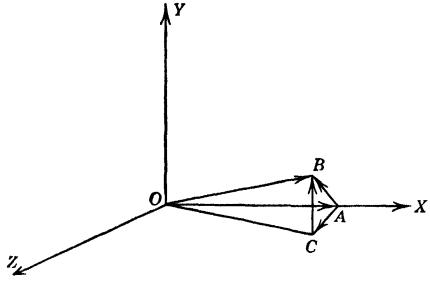


FIG. 6.33 The change of unit vectors with time.

larly **CB** represents a rotation about the Z axis in the time dt and is parallel to the Y axis, so that

$$\mathbf{CB} = \mathbf{j}\omega_z \, dt$$

where ω_z is the angular velocity of rotation about the Z axis. Substituting in Eqn. 6.53 gives

$$d\mathbf{i} = \mathbf{j}\omega_z \, dt - \mathbf{k}\omega_y \, dt$$

or

$$\frac{d\mathbf{i}}{dt} = \mathbf{j}\omega_z - \mathbf{k}\omega_y \tag{6.54}$$

Similarly it may be proved that

$$\frac{d\mathbf{j}}{dt} = \mathbf{k}\omega_x - \mathbf{i}\omega_z$$

and

$$\frac{d\mathbf{k}}{dt} = \mathbf{i}\omega_y - \mathbf{j}\omega_x$$

These latter two equations may be obtained from Eqn. 6.54 by cyclically rotating each of the items in the equation.

If the resultant angular velocity of rotation of the X, Y, Z axes relative to the X_0, Y_0, Z_0 axes is ω with the components ω_x, ω_y, ω_z along the X, Y, Z axes respectively as given above, then

$$\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$$

As an exercise in manipulating vectors you can readily prove from the above relationships that

$$\frac{d\mathbf{i}}{dt}x + \frac{d\mathbf{j}}{dt}y + \frac{d\mathbf{k}}{dt}z = \boldsymbol{\omega} \times \mathbf{r} \tag{6.55}$$

Hence we may now write Eqn. 6.51 as

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v} \tag{6.56}$$

and

$$\dot{\mathbf{r}}_0 = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v} \tag{6.57}$$

In words, these terms signify that:

$\dot{\mathbf{R}}$ is the velocity of the origin O of the moving axes relative to the origin O_0 of the fixed axes.

$\dot{\mathbf{r}}_0$ is the true velocity of the moving point P as measured relative to the fixed axes X_0, Y_0, Z_0.

$\dot{\mathbf{r}}$ is the velocity of the point P as would be measured by an observer who is located at the origin of the moving axes X, Y, Z and who is not rotating with the axes.

$\boldsymbol{\omega} \times \mathbf{r}$ is the velocity of the point P which is not moving relative to the origin O but whose motion is due to the rotation of the axes.

$\mathbf{v}$ is the velocity of the point P relative to the axes X, Y, Z or as recorded by an observer who is attached to, and rotating with, the moving axes X, Y, Z.

Thus the velocity of the point P which is given by the one term $\dot{\mathbf{r}}_0$ relative to the origin of the fixed axes is also given by three terms relative to the moving axes. These three terms are the motion of translation $\dot{\mathbf{R}}$ of the origin of the fixed set of axes relative to the origin of the moving set, $\boldsymbol{\omega} \times \mathbf{r}$ due to the rotation of the axes, and $\mathbf{v}$ due to the motion of the point P relative to the axes X, Y, Z which are now considered as stationary.

We next turn to the linear and angular accelerations referred to the fixed and moving axes. From Eqn. 6.57, by differentiation, the acceleration of the point P is given by

$$\ddot{\mathbf{r}}_0 = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\mathbf{v}} \tag{6.58}$$

Substituting for $\dot{\mathbf{r}}$ from Eqn. 6.56 gives

$$\ddot{\mathbf{r}}_0 = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathbf{v} + \dot{\mathbf{v}} \tag{6.59}$$

Differentiating Eqn. 6.52, we obtain for $\dot{\mathbf{v}}$:

$$\dot{\mathbf{v}} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z} + \dot{x}\dot{\mathbf{i}} + \dot{y}\dot{\mathbf{j}} + \dot{z}\dot{\mathbf{k}} \tag{6.60}$$

in which $\dot{\mathbf{i}}$ is used in place of $d\mathbf{i}/dt$, etc. Substituting from Eqn. 6.54 for $\dot{\mathbf{i}}$, $\dot{\mathbf{j}}$, $\dot{\mathbf{k}}$ and rearranging gives

$$\begin{aligned}\dot{\mathbf{i}}\dot{x} + \dot{\mathbf{j}}\dot{y} + \dot{\mathbf{k}}\dot{z} &= \mathbf{i}(\omega_y\dot{z} - \omega_z\dot{y}) + \mathbf{j}(\omega_z\dot{x} - \omega_x\dot{z}) + \mathbf{k}(\omega_x\dot{y} - \omega_y\dot{x}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{v}\end{aligned} \tag{6.61}$$

The first three terms in Eqn. 6.60 represent the apparent linear acceleration $\mathbf{a}$ of the point P relative to the moving axes X, Y, Z or

$$\mathbf{a} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}$$

Thus Eqn. 6.60 may be written

$$\dot{\mathbf{v}} = \mathbf{a} + \boldsymbol{\omega} \times \mathbf{v}$$

It is left as an exercise to prove that

$$\dot{\boldsymbol{\omega}} = \mathbf{i}\dot{\omega}_x + \mathbf{j}\dot{\omega}_y + \mathbf{k}\dot{\omega}_z$$

Thus the term $\dot{\boldsymbol{\omega}}$ may be considered to be the angular acceleration of the moving axes relative to the fixed axes X_0, Y_0, Z_0. Collecting the terms, we then have for the acceleration of the point P from Eqn. 6.58

$$\mathbf{a}_0 = \ddot{\mathbf{r}}_0 = \ddot{\mathbf{R}} + \mathbf{a} + 2\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{6.62}$$

As a summary we shall give the physical meaning of the terms in Eqn. 6.62:

$\mathbf{a}_0$ or $\ddot{\mathbf{r}}_0$: The acceleration of the point P relative to the fixed axes X_0, Y_0, Z_0 in the inertial system.

$\ddot{\mathbf{R}}$: The acceleration of the origin O of the moving axes X, Y, Z relative to the origin O_0 of the fixed axes X_0, Y_0, Z_0.

$\mathbf{a}$: The acceleration of the point P relative to the moving axes X, Y, Z.

$2\boldsymbol{\omega} \times \mathbf{v}$: The so-called *Coriolis acceleration*. It is the acceleration received by a point P moving relative to the axes which are rotating with the angular velocity $\boldsymbol{\omega}$.

$\dot{\boldsymbol{\omega}} \times \mathbf{r}$: The linear acceleration of a point P, at a distance $\mathbf{r}$ from the origin of the moving axes, due to the angular acceleration $\dot{\boldsymbol{\omega}}$ of these axes.

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$: The centripetal acceleration of the point P due to the rotation of the axes X, Y, Z. It is readily shown that this term is equal to $\omega^2 p$ where p is the perpendicular distance from the point P, or the head of the vector $\mathbf{r}$, to the angular velocity vector $\boldsymbol{\omega}$.

6.23 Motion of a Particle Falling toward the Earth

We have repeatedly emphasized that the earth is not an inertial system, though up to the present we have assumed that Newton's laws of motion are valid for a coordinate system attached to the earth. We shall now investigate how far we have been in error in making this assumption.

Let the center of the earth be the origin of a rotating set of axes rigidly attached to the earth. Relative to the origin O_0 of a set of axes X_0, Y_0, Z_0 in an inertial system, the acceleration of the center of the earth O is negligibly small. Thus we may set $\ddot{\mathbf{R}}$ in Eqn. 6.62 equal to zero. The angular velocity $\boldsymbol{\omega}$ of the earth is easily calculated to be about 7.3×10^{-5} radian per second. Since this angular velocity $\boldsymbol{\omega}$ is constant, it follows that its angular acceleration $\dot{\boldsymbol{\omega}}$ is zero.

Thus from Eqn. 6.62 the acceleration of a point P relative to the axes attached to the rotating earth is

$$\mathbf{a} = \mathbf{a}_0 - 2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{6.63}$$

Consider a particle falling toward the earth under gravity with the acceleration $\mathbf{g}$. Now when $\mathbf{g}$ is measured, as for example with a pendulum having a negligible velocity relative to the earth, its value is the resultant of the earth's gravitational attraction $\mathbf{a}_0$ and the centripetal acceleration $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. This effect was investigated in problem 20 of Chapter 1. Thus the value of the acceleration due to gravity is

$$\mathbf{g} = \mathbf{a}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{6.64}$$

From Eqns. 6.63 and 6.64 it follows that the acceleration $\mathbf{a}$ of a point moving with a velocity $\mathbf{v}$ on the earth is

$$\mathbf{a} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{v} \tag{6.65}$$

This acceleration is relative to the rotating axes X, Y, Z attached to the earth. Since the position of the point P on the earth, given by the vector $\mathbf{r}$, is not contained in Eqn. 6.65, it follows that the origin of the rotating axes may be placed anywhere on the earth and not only at the center of the earth. Let us take the origin of the rotating axes at some point O on the earth at latitude θ, Fig. 6.34. Suppose that the Y axis is drawn vertically upward and the Z axis on the horizon circle through O pointing north. Then for a right-handed system of

axes the X axis is directed perpendicularly out of the plane of the paper towards the west on the horizon circle.

Now suppose that a particle falls from rest from a point on the Y axis near the surface of the earth. Its velocity v after a time t is given approximately by

$$v = gt$$

and is directed toward the center of the earth approximately along the $-Y$ axis. For this particle the vector $(-2\omega \times \mathbf{v})$ is directed into

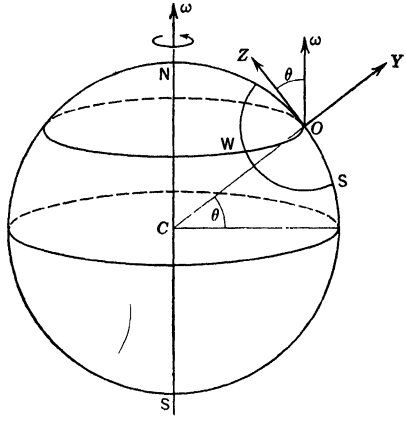


FIG. 6.34 Motion relative to the moving earth.

the plane of the paper, that is, along the $-X$ axis or in an easterly direction on the horizon circle through O. From Fig. 6.34 it can be seen that the magnitude of this vector $(-2\omega \times \mathbf{v})$ is

$$2v\omega \cos \theta = 2gt\omega \cos \theta \tag{6.66}$$

This then is the easterly acceleration of a particle dropped from rest above the surface of the earth. It signifies that a particle does not fall vertically down toward the earth or along a plumb line but falls east of this line. The direction of this easterly acceleration, given in Eqn. 6.66, is, by our convention, along the $-X$ axis. Hence we may write

$$-\ddot{x} = 2gt\omega \cos \theta$$

Since at time zero both the velocity $\dot{x}$ and the displacement x are zero, it follows that

$$-x = \frac{\omega g t^3 \cos \theta}{3}$$

This $-x$ is the easterly displacement of a particle starting from rest and falling for a time t under the constant gravitational acceleration g. It is the easterly displacement from the foot of a plumb line hung from the point where the particle is initially started. In this example it should be noticed that angular momentum is conserved.

6.24 Horizontal Deflection of Moving Air

If you have ever observed a weather map you have probably noticed that the winds are not from the high to the low-pressure regions but rather along the lines of equal pressure. This result is

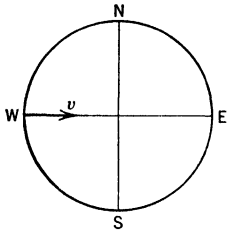


FIG. 6.35 Motion on a horizon circle.

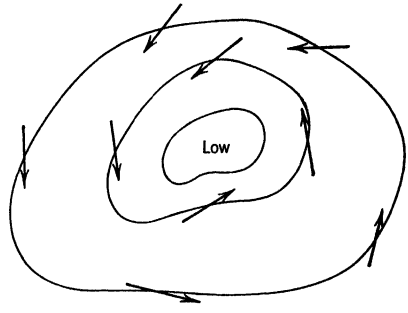


FIG. 6.36 The winds and curves of equal pressure in a cyclone.

given in some meteorological books as Buys Ballots' law, which states that if a person stands with his back to the wind the region of lowest barometric pressure is on his left hand and slightly towards his front. An explanation of this law is given in terms of the Coriolis acceleration.

Consider a westerly wind, that is, a horizontal mass of air moving from west to east with a velocity **v**, Fig. 6.35. In this horizon circle the effective angular rotation, from Fig. 6.34, is $\omega \sin \theta$, directed vertically upward out of the plane of the paper. The Coriolis acceleration $-2\boldsymbol{\omega} \times \mathbf{v}$ has a magnitude of $2\omega v \sin \theta$ and a direction toward the south in the northern hemisphere.

Thus a mass of air moving toward the west is deflected toward the south. It is easy to show from what has been given above that a low-pressure area gives rise to cyclones whose winds move in a counterclockwise direction, Fig. 6.36. In the southern hemisphere the winds would be in the opposite direction. A low-pressure area would give rise to winds in a clockwise direction.

6.25 Motion of Rigid Bodies Relative to a Fixed Point

The analogy between rotational and translational acceleration holds only when the rotation is about a fixed axis. For the translational motion of the center of mass of a rigid body the resultant force is equal to the product of the mass and the linear acceleration of the center of mass. Similarly for the rotational motion of a rigid body about a *fixed axis* the resultant torque is equal to the product of the moment of inertia about the axis and the angular acceleration of the body about the axis. In these cases mass and moment of inertia are both constants. The accelerations are then respectively proportional to the resultant force or the resultant torque. These accelerations, both linear and angular, are simple functions of the coordinates, either r or θ.

Now if the rotational motion is given with reference to a fixed point rather than with reference to a fixed axis, no simple relationship exists between the torque and the angular acceleration. This may seem incorrect since the torque referred to a fixed point is equal to the time rate of change of the angular momentum of the body about the point as given in Eqn. 6.19. Although this is a simple relationship, it yields no simple result, for the moment of inertia is a tensor and in general is not a constant quantity independent of the coordinates. In general for rotation about a fixed point it is not possible to have three angles whose time derivatives give the angular velocities about the axes from which these angles are measured. These difficulties would appear to make the solution of problems pertaining to rotation about a fixed point well nigh impossible. However, there are simplifications which considerably overcome these difficulties.

6.26 Moment of Inertia Tensor

Since the torque $\mathbf{L}$ exerted on a body rotating about a fixed point is equal to the time rate of change of angular momentum $\dot{\mathbf{\Omega}}$, we have from Eqn. 6.19

$$\vec{\mathbf{L}} = \dot{\vec{\mathbf{\Omega}}}$$

where from Eqn. 6.18 for the ith particle in the body, Fig. 6.15,

$$\mathbf{\Omega} = \sum_1^n m_i(\mathbf{r}_i \times \dot{\mathbf{r}}_i)$$

$$= \sum_1^n m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{6.67}$$

In this double vector product it is necessary first to evaluate the vector product $(\boldsymbol{\omega} \times \mathbf{r}_i)$. The vector product of $\mathbf{r}_i$ and the vector

$(\boldsymbol{\omega} \times \mathbf{r}_i)$ is then found. Notice that Eqn. 6.67 is not equivalent to the vector product $m_i(\mathbf{r}_i \times \boldsymbol{\omega}) \times \mathbf{r}_i$. As an exercise in algebraic manipulation you should expand Eqn. 6.67, using the determinant form for the vector product as in Eqn. 6.17 and also the following equations.

$$\mathbf{r}_i = \mathbf{i}x_i + \mathbf{j}y_i + \mathbf{k}z_i$$

$$\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$$

For the component of $\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)$ along the X axis you should find that

$$\mathbf{i}\Omega_x = \mathbf{i} \sum_1^n [\omega_x(y_i^2 + z_i^2) - \omega_y x_i y_i - \omega_z x_i z_i]\, m_i$$

The components along the Y and Z axes may be written from the above by cyclically changing the coordinates. Thus the angular momentum about a fixed point is

$$\boldsymbol{\Omega} = \mathbf{i}\Big[\omega_x \sum_1^n m_i(y_i^2 + z_i^2) - \omega_y \sum_1^n m_i x_i y_i - \omega_z \sum_1^n m_i x_i z_i\Big]$$

$$+ \mathbf{j}\Big[\omega_y \sum_1^n m_i(z_i^2 + x_i^2) - \omega_z \sum_1^n m_i y_i z_i - \omega_x \sum_1^n m_i y_i x_i\Big]$$

$$+ \mathbf{k}\Big[\omega_z \sum_1^n m_i(x_i^2 + y_i^2) - \omega_x \sum_1^n m_i z_i x_i - \omega_y \sum_1^n m_i z_i y_i\Big]$$

In this expansion it is usual to abbreviate the terms in the above equation in the following manner:

$$I_x = \sum_1^n m_i(y_i^2 + z_i^2) \qquad I_{xy} = -\sum_1^n m_i x_i y_i \qquad I_{xz} = -\sum_1^n m_i x_i z_i$$

$$I_y = \sum_1^n m_i(z_i^2 + x_i^2) \qquad I_{yz} = -\sum_1^n m_i y_i z_i \qquad I_{yx} = -\sum_1^n m_i y_i x_i$$

$$I_z = \sum_1^n m_i(x_i^2 + y_i^2) \qquad I_{zx} = -\sum_1^n m_i z_i x_i \qquad I_{zy} = -\sum_1^n m_i z_i y_i$$

These nine quantities I_x, I_{xy}, $\cdots$, I_z are the components of the moment of inertia of the body about the fixed X, Y, Z axes. The components I_x, I_y, I_z are usually called the moments of inertia, and the components I_{xy}, I_{yz}, $\cdots$, I_{zx} are called the products of inertia.

A quantity such as moment of inertia is called a tensor. As is seen Ω and ω do not necessarily have the same magnitude or direction.

Now it is possible to prove, though we shall not do so here, that any rigid body has three principal axes of inertia of such a nature that the products of inertia are zero when the principal axes are used as coordinate axes. This result is of considerable importance. In the following discussion we shall use the principal axes as the coordinate axes. For such axes the expression for the angular momentum becomes

$$\mathbf{\Omega} = \mathbf{i}I_x\omega_x + \mathbf{j}I_y\omega_y + \mathbf{k}I_z\omega_z \tag{6.68}$$

The kinetic energy of the rotating body may be written

$$T = \tfrac{1}{2}\sum_1^n m_i\dot{r}_i^2 = \tfrac{1}{2}\sum_1^n m_i(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i)$$

Using the principal axes and expanding, it follows that the kinetic energy is

$$T = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \tag{6.69}$$

6.27 Equations of Motion of a Rotating Rigid Body Referred to a Fixed Point

We shall now find the expressions for the components of the torque acting on a rigid body rotating about a fixed point in terms of the corresponding components of angular momentum. To do this let us suppose that the coordinate axes are the principal axes of the body and that these coordinate axes are rigidly attached to the body. The coordinate axes then rotate with the body. Thus the moments of inertia about each of the axes are constants. From Eqn. 6.19, $\mathbf{L} = \dot{\mathbf{\Omega}}$, and, using Eqn. 6.68 for $\mathbf{\Omega}$, it follows that

$$\mathbf{L} = \mathbf{i}I_x\dot{\omega}_x + I_x\omega_x\frac{d\mathbf{i}}{dt} + \mathbf{j}I_y\dot{\omega}_y + I_y\omega_y\frac{d\mathbf{j}}{dt} + \mathbf{k}I_z\dot{\omega}_z + I_z\omega_z\frac{d\mathbf{k}}{dt}$$

where $\dot{\omega}_x$ is the time rate of change of the angular velocity of the X axis and $d\mathbf{i}/dt$ is the time rate of change of the unit vector $\mathbf{i}$ along the rotating X axis. Substituting for $d\mathbf{i}/dt$, $d\mathbf{j}/dt$, $d\mathbf{k}/dt$ the values given in Eqn. 6.54, it follows that

$$\begin{aligned} L_x = \dot{\Omega}_x &= (I_x\dot{\omega}_x - I_y\omega_y\omega_z + I_z\omega_y\omega_z) \\ &= [I_x\dot{\omega}_x + \omega_y\omega_z(I_z - I_y)] \end{aligned}$$

and (6.70)

$$\begin{aligned} L_y = \dot{\Omega}_y &= [I_y\dot{\omega}_y + \omega_z\omega_x(I_x - I_z)] \\ L_z = \dot{\Omega}_z &= [I_z\dot{\omega}_z + \omega_x\omega_y(I_y - I_x)] \end{aligned}$$

The three equations given above for L_x, L_y, L_z are known as Euler's equations. They relate the torque about each one of the principal axes to the angular velocity of the principal axes. Since the principal axes are fixed in the rotating body, it follows that the moments of inertia I_x, I_y, I_z are constants independent of time. We shall now apply this theory to the gyroscope.

6.28 The Motion of a Symmetrical Rigid Body. The Gyroscope

In this section we shall analyze the motion of a gyroscope a little more rigorously than was done in section 6.21. This theory applies to the motion of any rotating symmetrical body. By a symmetrical body is meant one that has two of its principal moments of inertia equal to each other. These are generally perpendicular to a third axis, the axis of rotation of the body. The gyroscope, the earth, and certain types of atomic and molecular structure are examples of such symmetrically rotating bodies.

Let us consider the gyroscopic motion of a symmetrical spinning top. To describe this motion we shall choose the principal axes of the top as the rotating coordinate axes, X, Y, Z. These rotating axes have a common origin O with a set of axes X_0, Y_0, Z_0 which are fixed in an inertial system. The common origin of the axes is taken as the fixed point of the top. In Fig. 6.37 the Y axis is the spin axis of the top about the origin O of the coordinate systems. Any motion of the top, other than its spin, is measured in terms of the angles θ and ϕ. These angles are called the Eulerian angles and serve as independent variables in the equations of motion.

Let us first describe the motions of a spinning top. The top is spinning about the Y axis, which is rotating about the fixed Y_0 axis. In this motion the Y axis of the top describes a cone about the vertical Y_0 axis. This motion is known as the *precessional motion* and is measured by the time rate of change of the angle ϕ. In general the axis of the top changes its cone angle θ as the precession takes place. The angular velocity of θ is called *nutation*. Thus in general the top may be said to have three motions, spin, precession, and nutation.

The Eulerian angle ϕ is constructed in the following manner. The rotating OZ axis is perpendicular to the spin axis OY and also lies in the fixed X_0Z_0 plane. Thus the precessional angle ϕ is the angle between the fixed OZ_0 axis and the rotating OZ axis. To complete the system of rotating axes the OX axis is chosen perpendicular to the OY and OZ axes. From the nature of the construction it follows that OZ is perpendicular at every instant of time to the axes OY, OY_0, and OX. Thus the three axes OY, OY_0, and OX lie in the same plane.

We must try now to relate the Eulerian angles θ and ϕ to the components of the angular velocity of the rotating axes X, Y, Z. This we must do if we are to obtain the differential equations of motion of the top from Eqn. 6.70. Let ω be the vector angular velocity of rotation of the axes X, Y, Z with respect to the fixed axes X_0, Y_0, Z_0. The components of ω along the axes X, Y, Z are ω_x, ω_y, and ω_z respectively.

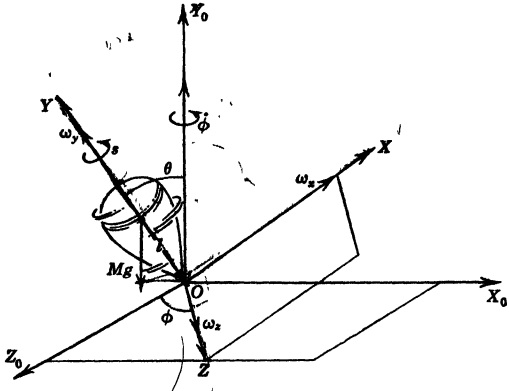


FIG. 6.37 The motions of a spinning top.

Since OZ is perpendicular to OY and OY_0, it follows that the component ω_z of the angular velocity about the OZ axis is measured by the time rate of change of the angle θ between the OY and OY_0 axis, or

$$\omega_z = \dot{\theta} = \frac{d\theta}{dt} \tag{6.71}$$

Again, since OY_0 is perpendicular to OZ_0 and OZ, it can be seen from Fig. 6.37 that the rate of change of ϕ, the angle between the OZ_0 and OZ axes, is along the fixed OY_0 axis. This angular velocity $\dot{\phi}$ about the OY_0 axis can be resolved into the components ω_x and ω_y along the rotating OX and OY axes. From Fig. 6.37 it may be seen that

$$\begin{aligned} \omega_y &= \dot{\phi} \cos\theta \\ \omega_x &= \dot{\phi} \sin\theta \end{aligned} \tag{6.71}$$

From the symmetry of the top the moments of inertia about the X and Z axes are equal, that is

$$I_x = I_z$$

Also the total angular velocity of the top about the Y axis is the sum of the spin angular velocity $\mathbf{s}$ and the angular velocity of the Y axis. This total angular velocity will be called $\mathbf{S}$.

$$\mathbf{S} = \boldsymbol{\omega}_y + \mathbf{s}$$

Now the only torque acting on the top is that due to its weight, and this acts about the Z axis.

$$L_z = Mgl \sin \theta$$

where l is the distance of the center of mass of the top from the point O. Also

$$L_x = L_y = 0$$

The equations of motion of the top are obtained from Eqn. 6.70, but now the component of angular momentum along the Y axis is $I_y(\omega_y + s)$ rather than $I_y\omega_y$ used in Eqn. 6.70. Hence, replacing I_z by I_x we have

$$\begin{aligned} &I_x\dot{\omega}_x + \omega_y\omega_z(I_x - I_y) - I_y s\omega_z = 0 \\ &I_y(\dot{\omega}_y + \dot{s}) + \omega_z\omega_x(I_x - I_z) = 0 \\ &I_x\dot{\omega}_z + \omega_x\omega_y(I_y - I_x) + I_y s\omega_x = Mgl \sin \theta \end{aligned} \tag{6.72}$$

From the middle equation above we have

$$I_y(\dot{\omega}_y + \dot{s}) = 0 \qquad I_y\dot{S} = 0 \qquad \text{or } I_yS = \text{Constant}$$

Since I_y is a constant, it follows that $\mathbf{S}$, the total angular velocity about the Y axis, is a constant.

Next we turn to the total energy U of the top. This is the sum of its kinetic and potential energies. Using Eqn. 6.69 and noting that the angular velocity about the Y axis is $(\omega_y + s)$ rather than ω_y, the total energy is

$$U = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y(\omega_y + s)^2 + \tfrac{1}{2}I_z\omega_z^2 + Mgl \cos \theta$$

Substituting for ω_x, ω_y, ω_z from Eqn. 6.71, we have

$$U = \tfrac{1}{2}I_x \sin^2 \theta\, \dot{\phi}^2 + \tfrac{1}{2}I_yS^2 + \tfrac{1}{2}I_x\dot{\theta}^2 + Mgl \cos \theta$$

We now require a further relationship between the angles θ and ϕ. This may be obtained by recognizing that there are no torques about the OY_0 axis and hence the component of angular momentum about this axis is a constant C.

$$\Omega_{Y_0} = I_yS \cos \theta + I_x\omega_x \sin \theta = C$$

or

$$C = I_y S \cos\theta + I_x \sin^2\theta\,\dot{\phi} \tag{6.73}$$

Thus

$$\dot{\phi} = \frac{C - I_y S \cos\theta}{I_x \sin^2\theta}$$

Substituting this value of $\dot{\phi}$ in the equation for the energy U given above, we obtain a relationship involving only θ and its derivatives. Thus

$$\dot{\theta}^2 = \frac{2}{I_x}\left[U - \frac{(C - I_y S\cos\theta)^2}{2I_x \sin^2\theta} - \frac{I_y S^2}{2} - Mgl\cos\theta\right] \tag{6.74}$$

Theoretically at least we should be able to determine the angular velocities of precession and nutation from Eqns. 6.74 and 6.73 respectively. However, Eqn. 6.74 is a difficult one to integrate and would take us much deeper into the analysis of gyroscopes than we wish to go. In order to avoid this we shall introduce a new variable p where

$$p = \cos\theta$$

and

$$\dot{p} = -\sin\theta\,\dot{\theta}$$

It follows from the definition of θ that p must lie between -1 and $+1$. Substituting for θ and $\dot{\theta}$ in Eqn. 6.74 gives

$$\begin{aligned}\dot{p}^2 &= \frac{2U}{I_x}(1-p^2) - \frac{(C - I_y S p)^2}{I_x^2} - \frac{I_y S^2(1-p^2)}{I_x} - \frac{2Mglp(1-p^2)}{I_x} \\ &= (1-p^2)\left(\frac{2U - I_y S^2}{I_x} - \frac{2Mglp}{I_x}\right) - \frac{(C - I_y S p)^2}{I_x^2} \\ &= (1-p^2)(A - Bp) - (D - Ep)^2 \end{aligned} \tag{6.75}$$

where A, B, D, E are the constants from the equation above. Equation 6.75 is a cubic and can vanish for no more than three values of p. The limits of the variable p are $+1$ and -1, and at these limits $\dot{p}^2$ is negative since

$$\dot{p}^2_{(p=1)} = -(D - E)^2 \qquad \text{and} \qquad \dot{p}^2_{(p=-1)} = -(D + E)^2$$

Thus the graph of $\dot{p}^2$ plotted against p must cross the p axis between -1 and $+1$ as shown in Fig. 6.38. In the physically possible motion the cone angle θ must lie between 0 and $\pi/2$ so that p must lie between 1 and 0. That is, the curve in Fig. 6.38 must cross the p axis between 1 and 0 rather than between -1 and $+1$. These values at which $\dot{p}^2$ is zero are called p_1, p_2. They correspond to two values θ_1 and θ_2 of

the cone angle and are called the *libration* limits. Notice that when p is very large compared to any of the constants then Eqn. 6.75 becomes

$$\dot{p}^2 = Bp^3$$

so that when p has a very large negative value $\dot{p}^2$ is negative, and when p has a very large positive value $\dot{p}^2$ is positive. It is from these considerations that Fig. 6.38 was qualitatively drawn.

This analysis shows that the cone angle θ varies between the libration limits θ_1 and θ_2 as the precessional motion takes place. In this

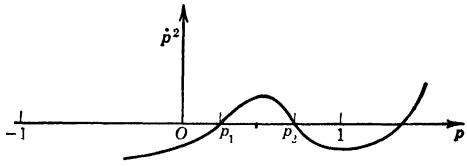


FIG. 6.38 Curve showing the variation of $\dot{p}^2$ with p where $p = \cos\theta$.

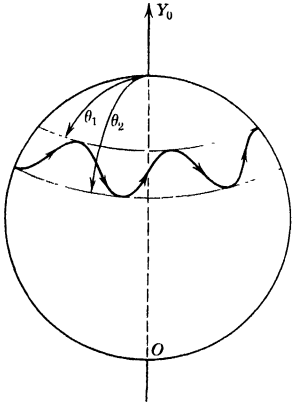


FIG. 6.39 Nutation of a top between the libration limits θ_1 and θ_2.

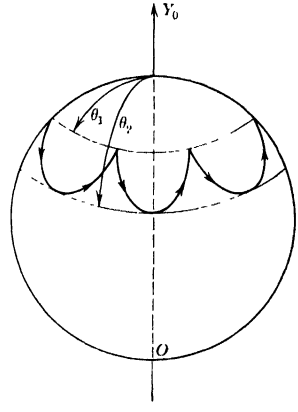


FIG. 6.40 The motion of a top started at a given angle.

motion both the angular velocities $\dot{\theta}$ and $\dot{\phi}$ vary with time. The expression for these variations is given in treatises exclusively devoted to this subject.

A possible type of motion for the axis of the top is shown in Fig. 6.39. Here the axis of the top moves between the libration limits θ_1 and θ_2 at the same time that the precessional motion about the fixed vertical axis OY_0 is taking place.

6.29 The Condition for Precession without Nutation

In section 6.21 we gave an approximate theory of the gyroscope. This applies to the special case where the spin angular momentum is very large compared with the precessional angular momentum and there is no nutational motion present. The top then precesses with its axis at a constant angle α with the fixed axis OY_0. In mathematical terms the conditions for this special case are

$$\theta = \alpha \qquad \dot{\theta} = 0$$

and from Eqn. 6.71

$$\omega_z = \dot{\theta} = 0 \qquad \omega_y = \dot{\phi} \cos \alpha \qquad \omega_x = \dot{\phi} \sin \alpha$$

Applying these conditions to the equation of motion, Eqn. 6.72, and setting $I_x = I_z$, we have for a top with a spin velocity s

$$\begin{aligned} I_x\dot{\omega}_x &= 0 \\ I_y(\dot{\omega}_y + \dot{s}) &= 0 \\ I_y(\omega_x\omega_y + s\omega_x) - I_x\omega_x\omega_y &= Mgl \sin \alpha \end{aligned}$$

These equations give

$$\dot{\omega}_y + \dot{s} = 0 \qquad \omega_y + s = \text{Constant} = S$$

and

$$I_y\omega_x S + I_x\omega_x\omega_y = Mgl \sin \alpha$$

or substituting for ω_x and ω_y and dividing by $\sin \alpha$ gives

$$I_y S\dot{\phi} - I_x\dot{\phi}^2 \cos \alpha = Mgl$$

Solving for $\dot{\phi}$ gives

$$\dot{\phi} = \frac{I_y S \pm \sqrt{I_y^2 S^2 - 4I_x \cos \alpha \, Mgl}}{2I_x \cos \alpha}$$

If $\dot{\phi}$ is to have real roots, i.e., the top precesses at a constant angle α without nutation, then

$$I_y^2 S^2 \geq 4I_x Mgl \cos \alpha \tag{6.76}$$

There are two possible rates of precession given by the positive and negative values of the square root. The larger of these two rates, obtained by taking the positive sign for the square root, is difficult to produce physically. Its precessional angular velocity is approximately

$$\dot{\phi}_1 = \frac{I_y S}{I_x \cos \alpha} \tag{6.77}$$

and is independent of the torque but dependent on the angle of inclination of the top.

The smaller angular precessional velocity ϕ_2 may be obtained by expanding the square root using the condition that $I_y^2S^2$ is very large compared to $4I_xMgl \cos \alpha$. This gives

$$\phi_2 = \frac{Mgl}{I_yS} \tag{6.78}$$

This was investigated in section 6.21. The angular velocity ϕ_2 given in Eqn. 6.78 shows that, the smaller the spin velocity, the larger the precessional velocity. However, this does not hold for small values of the spin velocity since the conditions of the problem require a large spin velocity according to Eqn. 6.76.

6.30 The Motion of a Top Started with a Spin at a Given Angle

In this section we shall investigate the motion of a top started with a given spin velocity s and with its axis making an angle α with the vertical. The initial conditions at zero time are

$$\theta = \alpha \qquad \dot{\theta} = 0 \qquad \dot{\phi} = 0$$

Thus at zero time the angular velocities ω_x, ω_y, ω_z of the rotating axes are zero. From Eqn. 6.73 the angular momentum about the vertical fixed axis OY_0 is a constant. Hence

$$I_yS \cos \alpha = I_yS \cos \theta + I_x \sin^2 \theta \, \dot{\phi}$$

or

$$\dot{\phi} = \frac{I_yS(\cos \alpha - \cos \theta)}{I_x \sin^2 \theta} \tag{6.79}$$

Assuming the total energy of the top to be a constant, we have from the expression for the energy U in section 6.28:

$$\tfrac{1}{2}I_x \sin^2 \theta \, \dot{\phi}^2 + \tfrac{1}{2}I_yS^2 + \tfrac{1}{2}I_x\dot{\theta}^2 + Mgl \cos \theta = \tfrac{1}{2}I_yS^2 + Mgl \cos \alpha$$

or

$$I_x(\sin^2 \theta \, \dot{\phi}^2 + \dot{\theta}^2) = 2Mgl(\cos \alpha - \cos \theta) \tag{6.80}$$

Substituting for $\dot{\phi}$ from Eqn. 6.79 gives

$$\dot{\theta}^2 \sin^2 \theta = \frac{2Mgl \sin^2 \theta}{I_x} (\cos \alpha - \cos \theta) - \frac{I_y^2S^2}{I_x^2} (\cos \alpha - \cos \theta)^2$$

For convenience we shall introduce a new constant γ where

$$\gamma = \frac{I_y^2S^2}{4I_xMgl}$$

Then

$$\dot{\theta}^2 \sin^2 \theta = \frac{2Mgl \sin^2 \theta}{I_x} (\cos \alpha - \cos \theta) - \frac{4Mgl\gamma}{I_x} (\cos \alpha - \cos \theta)^2$$

From this equation, $\dot{\theta} = 0$ when

$$\theta = \alpha$$

and also when

$$\sin^2 \theta - 2\gamma(\cos \alpha - \cos \theta) = 0$$

or when

$$\cos^2 \theta - 2\gamma \cos \theta + 2\gamma \cos \alpha - 1 = 0$$

Thus $\dot{\theta}$ is zero when

$$\cos \theta = \gamma \pm \sqrt{\gamma^2 + 1 - 2\gamma \cos \alpha}$$

Both roots of this equation are real, but the larger of the two roots is greater than unity and corresponds to a physically impossible situation. The smaller root, involving the negative value of the square root, must be used. Hence the values of θ at which the nutational velocity is zero are

$$\theta_1 = \alpha \quad \text{and} \quad \cos \theta_2 = \gamma - \sqrt{\gamma^2 + 1 - 2\gamma \cos \alpha}$$

When the spinning top is initially set free at an angle α with the vertical, the top begins to descend and then the precessional and nutational motions set in. The axis of the top varies between the two values $\theta_1 = \alpha$ and the value of θ_2 given above. At the angle α, the values of the precessional and nutational motions are zero. Hence the axis of the top makes successive cusps with the horizontal circle around $\theta = \alpha$ as shown in Fig. 6.40. At the other libration limit where $\theta = \theta_2$ the nutational angular velocity $\dot{\theta}$ is zero and the precessional angular velocity may be obtained by dividing the energy equation, Eqn. 6.80, by the angular momentum equation, Eqn. 6.79, and setting $\dot{\theta} = 0$. This gives

$$\frac{I_x \sin^2 \theta \, \dot{\phi}^2}{I_x \sin^2 \theta \, \dot{\phi}} = \frac{2Mgl(\cos \alpha - \cos \theta)}{I_y S(\cos \alpha - \cos \theta)}$$

or at $\theta = \theta_2$

$$\dot{\phi} = \frac{2Mgl}{I_y S}$$

Thus the upper end of the top touches the horizontal circle of angle θ_2 as shown in Fig. 6.40.

6.31 The Gyroscopic Compass

Possibly the most widespread application of the gyroscopic principle is in the gyroscopic compass. This is used for navigation on many ships and airplanes and supplements the older magnetic compass. The theory of the gyroscopic compass was originally given by the French physicist Foucault. In its simplest form such a compass consists of a rapidly rotating circular disk whose axis is free to turn in a

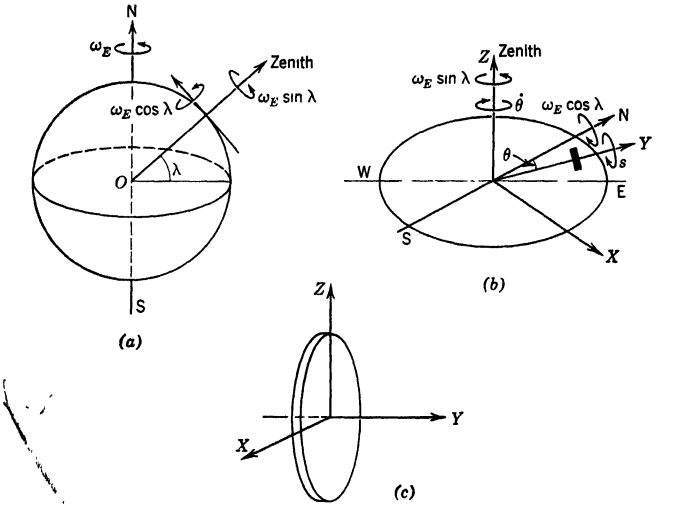


FIG. 6.41 The gyroscopic compass. (*a*) The motions on the earth. (*b*) The horizon circle. (*c*) The axes of the spinning wheel.

horizontal plane. We shall prove that due to the earth's motion the system is in equilibrium when the axis of the rotating disk lies along a geographical north-south line or along a meridian.

Suppose that the earth is rotating on its axis with a constant angular velocity ω_E. At a place on the earth at north latitude λ this angular velocity may be resolved into components $\omega_E \sin\lambda$ along the vertical and $\omega_E \cos\lambda$ about the horizontal, in a northerly direction, Fig. 6.41*a*. Let the angular velocity of spin of the disk be **s**. Take a system of axes X, Y, Z attached to the disk with the origin at the center of the disk, and let the Y axis be the axis of spin as in the case of the top. The X, Y, Z axes are as shown in Fig. 6.41*b* and *c*. Suppose that originally the Y axis of the disk makes an angle θ in the horizon circle

with the meridian pointing north and that this angle is positive when measured east of the north line as shown in Fig. 6.41*b*. Thus the rotational velocity of the axis of the disk is measured by $\dot{\theta}$ along the negative Z direction or by $-\dot{\theta}$ along the positive Z axis. Owing to the rotation of the earth there are components of angular velocity ω_x, ω_y, ω_z along the X, Y, Z axes respectively. From Fig. 6.41 these are

$$\omega_x = -\omega_E \cos \lambda \sin \theta$$

$$\omega_y = \omega_E \cos \lambda \cos \theta$$

$$\omega_z = \omega_E \sin \lambda - \dot{\theta}$$

If the axis of the disk is to place itself along a meridian, this implies motion about the vertical Z axis. We shall not need to have the X and Z axes rotate with the disk in order for their moments of inertia to be constant since the disk is symmetrical. Hence we can consider the Z axis to remain vertical and the components of angular velocity ω_x, ω_y, ω_z to be due to the rotation of the earth. The torque about the Z axis is given in Eqn. 6.70 with the modification that here the angular momentum about the Y axis is $I_y(\omega_y + s)$ rather than $I_y\omega_y$ as in Eqn. 6.70. Since there is no torque about the Z axis and $I_z = I_x$, we have

$$I_x\dot{\omega}_z + \omega_x\omega_y(I_y - I_x) + s\omega_x I_y = 0$$

or by substitution

$$-I_x\ddot{\theta} - \omega_E{}^2 \cos^2 \lambda \sin \theta \cos \theta(I_y - I_x) - I_y s\omega_E \cos \lambda \sin \theta = 0$$

Since the angular velocity of the earth ω_E is approximately 7.3×10^{-5} radian per second, it is negligibly small compared to the angular velocity of spin of the disk. Thus, without making any serious error, the second term containing $\omega_E{}^2$ may be neglected. It then follows that

$$\ddot{\theta} + \frac{(I_y s\omega_E \cos \lambda)}{I_x} \sin \theta = 0$$

For small values of the angle θ we may replace $\sin \theta$ by θ, and for this the motion of the axis of the disk is simple harmonic in character about the geographic meridian. From Eqn. 5.5 the period of the motion is

$$P = 2\pi \sqrt{\frac{I_x}{I_y s\omega_E \cos \lambda}}$$

The torque on the axis causing it to swing into the meridian is a maximum at the equator where λ is zero and zero at the poles. In

practice the motion of the gyroscopic compass is damped so that the oscillations about the meridian do not continue indefinitely. A gyroscopic compass has to be constructed exceedingly carefully if it is to operate successfully. The next time you are on a ship you should see this gyroscope if you can, for then you will appreciate how complicated such a compass can be.

PROBLEMS

1. Masses of 5, 6, and 9 gm are located at the respective points A, B, C of an equilateral triangle whose sides are 10 cm long. Find the x and y coordinates of the center of mass from A, if A is considered as the origin and AB as the X axis.

2. The density of a rod increases uniformly along its length from 1 to 10 gm per cm. Find the total mass of the rod of length 100 cm, and the distance of the center of mass from the end having the smallest density.

3. A uniform circular disk whose radius is 15 cm has a circular area 7.5 cm in radius cut out between the periphery and the center of the 15-cm disk. Find the distance of the center of mass from the center of the 15-cm disk.

4. A solid, uniform, right circular cone has a vertical height of 20 cm and a base whose radius is 10 cm. If the upper 10 cm of the cone is cut away, find the height of the center of mass above the base of this truncated portion.

5. The mass of the moon is about 0.013 times the mass of the [redacted] and its distance from the center of the earth is about 60 times the [redacted] the earth. If the radius of the earth is 4000 miles, find the distance of the center of mass of the earth-moon system from the center of the earth.

6. According to Newton's law of gravitation the earth and moon attract each other and are in turn attracted by the sun. What is the path of the center of mass of the earth-moon system about the sun, and what are the paths of the earth and moon about their own center of mass?

7. A uniform meter stick whose mass is 150 gm lies at rest on a horizontal table. Two small masses of 150 gm and 200 gm are placed at the 0-cm mark and the 100-cm mark respectively of the stick. A resultant force of 1000 dynes acts continuously at right angles to the stick at the 40-cm mark. Find the center of mass of the stick and the distance through which the center of mass moves in 10 sec. Does the answer to the second part of the question depend on where the force is applied to the stick?

8. Two particles A and B, initially at rest, have masses of 200 gm and 500 gm respectively and are 140 cm apart. They attract each other with a constant force of 2000 dynes. Find the positions of particles A and B after 3 sec, and find how much the center of mass has moved during the 3 sec. At what distance from the initial position of A do the particles collide?

9. Two particles A and B are placed along the X axis: A, whose mass is 50 gm, at the origin, and B, whose mass is 150 gm, 100 cm from the origin. The initial velocity of A is 10 cm per sec along the negative Y axis, and that of B is 20 cm per sec at an angle of 45° with the positive X axis. The particles attract each other with such a force that the particle B is at its starting point after 10 sec. Find:

a. The center of mass of the system at zero time.

b. The x and y components of the velocity of the center of mass at zero time.

c. The x and y coordinates of the center of mass after the 10 sec.

d. The x and y coordinates of the 50-gm mass after the 10 sec.

10. Two particles A and B are placed along the X axis; A, whose mass is 50 gm, at the origin; B, whose mass is 150 gm, 100 cm from the origin. The initial velocity of A is 10 cm per sec at an angle of 60° with the positive X axis, and that of B is 10 cm per sec along the positive Y axis. If the particles attract each other and collide after 5 sec, find the coordinates of the point of collision

11. Prove from fundamental principles that the results given in the table of moments of inertia, section 6.12, are correct

12. A 100-lb wheel 1 ft in radius has a concentric axle attached to it which has a radius of 3 in. and a weight of 16 lbw. The wheel and axle are free to rotate about the common axis. A light cord is wound around the axle, and a 5-lb weight hangs vertically from the end of the cord. Find the acceleration of the hanging weight and also the tension in the cord.

13. A circular disk having a radius of 2 ft and weighing 40 lbw has a small weight of 20 lbw attached to it at a distance of 1.5 ft from its center. If the disk is set in rotation about an axis through its center so that its minimum speed is 120 rpm, find its maximum speed.

14. A heavy flywheel is rotating on its axis and is slowing down owing to the friction of its bearings. At the end of the first minute its angular velocity is 0.8 of its angular velocity ω_0 at the beginning of the minute Find the ratio of its angular velocity at the end of the second minute to the angular velocity ω_0 if the moment of the frictional forces is (*a*) constant; (*b*) proportional to the angular velocity.

15. If **A, B, C** are three vectors having a common origin, show that $(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})$ represents the volume of the parallelopiped formed by these three vectors. Show that the volume of the parallelopiped is also given by $(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})$. Show analytically, by expressing the vectors in terms of their components along the X, Y, Z axes, that

$$(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})$$

16. A uniform plane circular disk whose mass is 500 gm and whose radius is 10 cm is free to turn in a vertical plane about a horizontal axis through its center. Suppose that the disk has a small light projection attached to its periphery and that originally the disk is at rest with the projection placed vertically above the center of the disk. A piece of wax whose mass is 100 gm is thrown horizontally with a velocity of 100 cm per sec so that it becomes attached to the projection. Find the resulting angular velocity of the disk and wax.

17. A so-called Borda's pendulum is made up of a spherical bob of radius a and a long thin wire of negligible weight If the length of the wire from the point of suspension to the center of the bob is l, show that the period of the pendulum for small oscillations is

$$P = 2\pi \sqrt{\frac{l}{g}\left(1 + \frac{2}{5}\frac{a^2}{l^2}\right)}$$

If this pendulum is considered as a simple pendulum, show that the period calculated on this basis is in error by about two parts in a hundred thousand when the radius of the spherical bob is one-hundredth of the length of the wire.

18. A uniform rod of mass M and length $2h$ is suspended from the same horizontal level by two vertical cords of length l which are attached to its ends. Show that the period of this bifilar suspension for small oscillations about a vertical axis through the center of the rod is

$$P = 2\pi \sqrt{\frac{Il}{Mgh^2}}$$

where I is the moment of inertia of the rod about a vertical axis through its center of mass.

19. A steel sphere of radius r executes simple harmonic oscillations of small amplitude on a concave surface of radius R. Show that the period of the oscillations P is given by

$$P = 2\pi \sqrt{\frac{1.4(R - r)}{g}}$$

(Hint: Set up the equation of motion from the energy equation.)

20. A ring of inner radius r_1, outer radius r_2, and mass m has a rectangular cross section. This ring is suspended from a knife edge perpendicular to the plane of the disk. Find the period of the ring for small oscillations about the knife edge.

21. A compound pendulum has a length AB of 100 cm and a period of 1.50 sec when allowed to oscillate in a vertical plane about an axis at one end A and a period of 1.51 sec when oscillating about an axis through the end B. Find the distance of the center of mass from the end A and the radius of gyration of the pendulum about an axis through its center of mass.

22. A Kater's pendulum has been carefully adjusted so that the period about one knife edge A is almost the same as that about the other knife edge B. The distance between the knife edges as measured by a comparator is 99.8503 cm. The periods are obtained by the method of coincidences, using an accurate 2-sec pendulum. It is found that for 500 half-oscillations about knife edge A there are 501 half-oscillations of the 2-sec pendulum, whereas after 10,000 half-oscillations about knife edge B there are 10,021 half-oscillations of the 2-sec pendulum. From an approximate measurement the center of mass is 53.9 cm from the knife edge B and 45.9 cm from A. From these data determine the value of the acceleration of gravity g.

23. The determination of the constant of gravitation G was made by an apparatus similar to that shown in Fig. 4.1. Two small spheres, each having a mass of 10 gm, were placed on the ends of a light rod 10 cm long. The light rod was supported horizontally from its center by a long thin vertical fiber. The period of torsional oscillation of this system was 769.2 sec. A fixed sphere whose mass is 10 kgm was placed near each suspended sphere so as to produce the maximum torsion. It was found that there was an angular deflection of the suspended rod of 0.02 radian when the distance between the centers of the large and small spheres was 10 cm. From this data calculate the constant of gravitation G.

24. A sphere and a cylinder, each having the same mass and radius, roll without slipping down an inclined plane whose inclined length is l and whose angle of inclination is θ. If both sphere and cylinder start at the same time from rest at the top of the plane, find:

a. Which reaches the bottom of the plane first.

b. Which has the larger translational kinetic energy at the bottom of the plane.

c. Which has the larger total energy at the bottom of the plane.

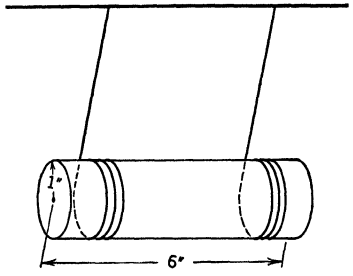
25. A sphere rolls up a rough inclined plane having an angle of inclination of 30°. If the translational velocity of its center of mass at the bottom of the plane is 20 ft per sec, find how far the sphere travels up the plane and the time it takes to reach the bottom again.

26. An 8-lbw block is placed on a plane inclined at an angle of 30°. A cord parallel to the plane, passing over a pulley with a mass of 4 lbw and a radius of 6 in. at the top of the plane, attaches the block to a hanging weight of 20 lbw. The coefficient of friction between block and plane is 0.2.

a. Find the acceleration of the block up the plane, and the tension in the cord on each side of the pulley.

b. If the hanging weight is changed to 5 lbw but otherwise the system remains the same, find in what direction the system moves

27. A cylindrical rod 1 in. in radius, 6 in. long, and weighing 4 lb has two cords wound around it whose ends are attached to the ceiling as shown in the figure.



PROBLEM 27

The rod is held horizontally with the two cords vertical. When the rod is released, the cords unwind and the rod rotates. Find the translational acceleration of the rod and the tension in the cord as the rod falls vertically.

28. A 4-lb weight hangs from one end of a cord that passes over a pulley that is free to rotate about its horizontal axis and has a radius of 3 in. and a weight of 2 lbw. The other end of the cord is wound around a cylinder whose radius is 4 in. and whose weight is 16 lbw. When released, the 4-lb weight accelerates upward and the cylinder downward as the string unwinds. Find the upward acceleration of the hanging weight and the angular acceleration of the cylinder.

29. A fixed pulley 3 in. in radius and weighing 2 lbw is free to rotate about a frictionless horizontal axis through its center. A cord passes over the pulley, the left side of the cord being wrapped around a cylinder A, 4 in. in radius and weighing 4 lbw. The right side of the cord is wrapped around a cylinder B, having a radius of 6 in. and a weight of 6 lbw. Originally the cylinders are at rest with the cords vertical. When released the cords unwrap, causing the cylinders to rotate. Find the angular acceleration of the pulley and the linear accelerations of the centers of mass of the cylinders A and B relative to the fixed axis through the pulley.

30. A meter stick held vertically with one end resting on a table is allowed to fall over. Find the velocity of the far end when it hits the table, assuming that the end on the table does not slip.

31. Find the reaction of the table on the end of the meter stick in problem 30 just before it strikes the horizontal table if the meter stick weighs 150 gm.

32. A meter stick weighing 150 gm lies on a frictionless horizontal table and is free to rotate about a vertical axis through one end. It is struck a blow whose

impulse is 10,000 dyne sec at a point 80 cm from the pivoted end and at right angles to the length of the rod. Find the kinetic energy of the rod and the impulse on the axis.

33. A meter stick weighing 150 gm lies on a horizontal frictionless table and is subject to no constraints. It is struck a blow whose impulse is 10,000 dyne sec at the 80-cm mark in a direction at right angles to the length of the rod. Find the point about which the rod begins to turn and the kinetic energy received by the rod.

34. A man holding a weight in each hand with his hands at his sides stands on a horizontal platform that can rotate about a frictionless vertical axis. The platform and man originally have a moment of inertia of 2 slugs ft^2 and an angular velocity of 1.8 revolutions per second. The man raises the weights by stretching his arms out and changes the moment of inertia of the system to 6 slugs ft^2. Calculate the new angular velocity of the system and the change in kinetic energy. Explain how this change in kinetic energy is brought about.

35. A meter stick whose mass is 150 gm rests on a frictionless horizontal table and is free to turn about a smooth pivot at its end or zero centimeter mark. Almost in contact with the rod at the 40-cm mark is an inelastic piece of putty having a mass of 50 gm. A horizontal blow whose impulse is 10^4 dyne sec is given to the meter stick at the 80-cm mark in a direction perpendicular to the rod. Find the resulting angular velocity of the rod and the impulsive actions at the pivot and on the putty.

36. A uniform rod of mass 240 gm, having one end attached to a smooth hinge, is allowed to fall vertically from a horizontal position. Show that the horizontal force on the hinge is a maximum when the rod makes an angle of 45° with the vertical and that the vertical force on the hinge at this angle is 330 gmw.

37. A plane circular disk having a weight of 6 lbw is free to turn about a horizontal axis through a point O at its circumference. The axis is perpendicular to the plane of the disk. The motion starts when the diameter through O is vertically above O. Prove that when this diameter has turned through an angle θ the components of the force on the axis along, and perpendicular to, this diameter are respectively $(14 \cos \theta - 8)$ lbw and $2 \sin \theta$ lbw.

38. A pendulum is made up of a solid sphere of radius R attached to a rod of length l. The system is suspended on a horizontal axis through the free end of the rod. The masses of the sphere and rod are equal. Show that there is no force on the axis if the pendulum is struck at a distance of

$$[\tfrac{4}{5}R^2 + \tfrac{8}{3}l^2 + 4Rl]/(3l + 2R)$$

39. A uniform rod AB weighing 25 lbw and having a length of 10 ft hangs vertically from a smooth horizontal axis at A. The rod is struck normally at a point 8 ft below A by a blow whose impulse is 120 lbw sec. Find the instantaneous angular velocity of the rod and the impulse received by the axis.

40. A uniform solid cylinder rolls down a plane whose angle of inclination is θ. Show that, if there is to be rolling and no sliding, the coefficient of friction between the cylinder and the plane must be equal to or greater than $\frac{1}{3} \tan \theta$.

41. A sphere whose radius is 10 cm and whose mass is 200 gm starts from rest from the top of an inclined plane whose length is 100 cm and whose angle of inclination is 30°.

a. Show that there is slipping if the coefficient of friction μ between the sphere and the plane is less than $\frac{2}{7} \tan 30°$.

Now suppose that the coefficient of friction is 0.1 so that there is both rolling and slipping.

b. Show that the velocity of the point of contact of the sphere on the plane after a time t is gt (sin 30° − 3.5μ cos θ).

c. Show that the time taken for the sphere to reach the bottom of the plane is 0.703 sec.

d. Find the linear velocity of the center of mass of the sphere at the bottom of the plane.

e. Find the angular velocity of the sphere at the bottom of the plane.

f. Find the translational and rotational kinetic energy of the sphere at the bottom of the plane.

g. Find the energy expended against friction by the sphere in moving down the plane.

42. A uniform rod having a mass of 200 gm and a length of 100 cm is lying on a smooth horizontal table and is struck horizontally at one end by a blow of 10,000 dyne sec in a direction perpendicular to its length. Find the linear velocities with which the two ends of the rod begin to move.

43. An airplane has a total moment of inertia about an axis perpendicular to the plane of rotation of the propellor of 20 slugs ft^2. The propellor has an angular velocity of 3600 rpm. If the airplane moves in a horizontal circle of 200-yd radius with a speed of 180 mph, find the torque exerted on the airplane and the direction of this torque.

44. A gyroscope is constructed from a plane circular disk 5 cm in radius and 800 gm in mass capable of rotating about an axle 200 gm in mass and 50 cm in length, as illustrated in Fig. 6.31. The axle is supported at its far end on a vertical pivot and the disk set in rotation with an angular velocity of 2106 rad/sec. If the axle is originally horizontal, show the angular velocity of precession is 20 rpm.

45. A circular wheel whose radius of gyration is 5 cm and whose mass is 500 gm is free to rotate on one end of a light axle whose length is 50 cm. At the other end of this axle a weight of 1 kg is placed. The system is supported on a vertical pivot 30 cm from the wheel. The wheel is rotating at an angular velocity of 10 revolutions per second. If the axle is originally horizontal, find the angular velocity of precession.

46. A top such as that shown in Fig. 6.37 has the moments of inertia I_x equal to 0.5 lb ft^2 and I_y equal to 0.1 lb ft^2. The weight of the top is 2 lbw, and the distance l of the center of mass from the point of rotation is 6 in. Show that steady motion is possible with the axis OY inclined at 30° to the vertical if the spin is greater than about 71 radians per second.

47. A top is made by forcing a light pin through the center of a uniform circular disk whose radius is 2 in. The pin projects 3 in. below the disk. If the top is set into steady motion so that the rim of the disk just fails to touch the ground, find the minimum number of revolutions per second of the disk.

48. An axle is placed through the center of a uniform circular disk so that the axle is inclined to the plane of the disk at an angle α. The system rotates with an angular velocity ω. Show that there is a torque on the bearings holding the axle which has a magnitude of $(I_y - I_x)\omega^2 \sin \alpha \cos \alpha$. This torque lies in a plane containing the axle and the perpendicular to the disk. The principal moments of inertia I_y, I_x are discussed in section 6.26.

49. The axis of a gyroscope is mounted so that it is always horizontal but the frame containing the gyroscope can turn about a vertical axis. This system forms a simple gyroscopic compass. If a torque L is applied to the frame about a vertical axis show that the axis of the gyroscope comes to rest at an angle θ with the meridian where

$$\sin \theta = \frac{L}{I_y \omega_E s \cos \lambda}$$

The notation is that used in section 6.31.

50. A body is dropped from the top of a tower 150 ft high at a latitude of 40° N. Find the easterly displacement of the point directly below the point of dropping. Neglect air resistance but take into account the rotation of the earth.

51. A body is projected vertically upward with an initial velocity v at a latitude λ. Show that the distance between the point of projection and the point of return to the ground is $1.33\omega v^3 \cos \lambda / g^2$. Is the point of return east or west of the point of projection? In the above, ω is the angular velocity of the earth and air resistence is neglected.

7 · ELASTICITY

7.1 Hooke's Law

In the previous chapters we have been concerned with the action of forces on rigid bodies. These bodies have been translated or rotated as a whole, but with the exception of the spring they have not been considered as changing in size or shape. Any changes in size or shape were of secondary importance, and their effects were assumed to be zero. We shall now drop this assumption.

Any change from the normal configuration of a body in either size or shape is a deformation, and the corresponding fractional change is a *strain*. In general such a change is the result of a force that acts either on the surface or on the body as a whole. If a force ΔF acts on area ΔA at some point of a body, then the *stress* at the point within this area is

$$\text{Stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

Where the force F is uniformly distributed over an area A, then the stress over the area A is F/A.

It is necessary in discussing stress-strain relationships in different materials to divide these into two main classes called *isotropic* and *anisotropic*. In isotropic materials the strain at any point is independent of direction, or the elastic properties of the material are the same in every direction. For anisotropic materials the elastic properties at any point are different in different directions about the point. Quartz and many other crystalline substances are anisotropic. They also show different optical properties in different directions. The analysis of the relationship between stress and strain for anisotropic materials is relatively complicated and must be left to more advanced treatises. Here we shall limit the discussion to isotropic materials. This limitation, although considerably simplifying the analysis, still permits us to discuss many important phenomena. We shall also assume that the materials are homogeneous, that is, they have the same elastic properties at all points in the material. If this were not assumed, the analysis would be impossible. Thus the analyses given in this chapter apply only to *homogeneous isotropic materials*. Further-

more, unless the stress is relatively small the strain is not proportional to it. Within the limits in which this proportionality holds, the modulus of elasticity is defined as the ratio of the stress to the corresponding strain. This region of proportionality is usually designated as the region in which Hooke's law is valid. Robert Hooke (1635–1703) showed experimentally for coiled and flat springs and the bending of wooden rods that if "one power stretch or bend it one space, two will bend it two and three will bend it three, and so forward."

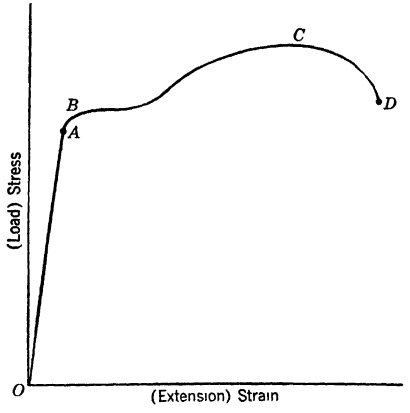


FIG. 7.1 Stress-strain curve for structural steel.

In order to investigate the limitations of Hooke's law let us consider the stretching of a long, uniform bar. Suppose that such a bar is hung vertically from a fixed support and has increasing loads placed on its lower end. In this bar the stress is measured by the load or force divided by its area of cross section and the strain by the increase in length divided by its original length. Thus, if the cross-sectional area is A and the load F, then

$$\text{Stress} = \frac{F}{A}$$

and, if the bar of original length l is elongated an amount Δl by the force, then

$$\text{Strain} = \frac{\Delta l}{l}$$

The shape of the stress-strain curve depends very much on the nature of the material composing the bar. Figure 7.1 shows such a

curve for structural steel though the curves change quite markedly with the carbon content of the steel. As the stress is initially increased, there is a corresponding proportional increase in the strain up to some point, A, Fig. 7.1, called the *proportional limit.* For structural steel the proportional limit is about 30,000 psi (lb per sq. in.). Since there is some departure from strict proportionality near the point A, it is common to define the proportional limit as that point on the stress-strain diagram at which there is 0.1% deviation from the straight line. In general all structures are made so that the strains are well below the proportional limit, and in all the work of this chapter we shall assume this to be so, i.e., that Hooke's law is valid.

Another term of importance in this discussion is *elastic limit.* This is defined as the maximum stress a material can withstand without a permanent elongation or set remaining after the stress is completely removed. For many materials the elastic limit and the proportional limit are practically the same.

For a load greater than that corresponding to the proportional limit the elongation of the bar increases rapidly as shown at B in the figure. There is then plastic flow in the material of the bar. For some materials there is an elongation even when the load is reduced. The stress corresponding to the point B is called the *yield point.* With further elongation of the bar there is again a region in which the stress increases with the strain. There is some point C, the maximum of the curve, at which the stress has its maximum value. This maximum stress the bar is capable of withstanding is called the *ultimate strength* of the material. After the conditions corresponding to the point C are reached, the bar elongates even if the load is decreased. Finally some point D is reached that corresponds to fracture of the bar.

The rate at which a body attains its unstrained or normal configuration after the stress is removed differs for different materials and may also differ for the same material, depending on its previous treatment. The rate of attainment of the equilibrium position is important in many instruments in which the restoring torque is supplied by a fine fiber. It is found that in this respect quartz fibers are superior to glass fibers and fine platinum fibers are intermediate between these two. Professor C. V. Boys used a quartz fiber for suspending moving masses in the Cavendish experiment for determining the gravitational constant G. Quartz fibers are also used in many electrometers, two of the more common being the Lindemann and Lauritsen electrometer. In all these the fiber must never be twisted beyond its elastic limit. Another phenomenon, discovered by Lord Kelvin, is called elastic fatigue. He found that the rate at which the oscillations in a wire

died away was slower when the wire had been previously oscillated. The wire behaved as though it were fatigued and, like us, recovered its normal properties after a rest. These and other phenomena pose some difficult problems for the metallurgist who tries to explain them in terms of crystal forces and other fundamental concepts.

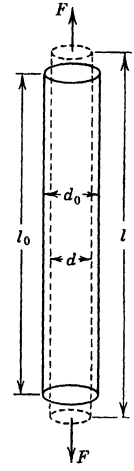


FIG. 7.2 Lateral contraction accompanying longitudinal extension.

7.2 Young's Modulus E and Poisson's Ratio σ

This modulus of elasticity E is concerned with the stretch produced by a simple tension. A convenient method of measuring Young's modulus for a substance is to prepare a long wire of uniform cross section from some of the substance. This wire is rigidly held at its upper end, and its elongation is measured for various loads placed on the lower end. The strain is measured by the increase in length divided by the original length, and the stress by the stretching force divided by the area of cross section of the wire. If a wire of length l and cross section A is subjected to a force F that produces an elongation Δl, then Young's modulus for the material of the wire is

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{A} \cdot \frac{l}{\Delta l}$$

From this equation it follows that the dimensions of Young's modulus are $[ML^{-1}T^{-2}]$ in the absolute system or $[FL^{-2}]$ in the gravitational system. The units in which moduli of elasticity are usually measured are dynes per square centimeter or pounds weight per square inch. For most common metals Young's modulus is of the order of 10^{12} dynes per cm^2.

When a wire is extended there is in general an accompanying decrease in the area of cross section. In Fig. 7.2 is shown an exaggerated picture of this phenomenon. A wire of original length l_0 and diameter d_0 is subjected to a tension F changing the length to l and the diameter to d. The longitudinal strain is

$$\frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$$

and the transverse strain is

$$\frac{d - d_0}{d_0} = -\frac{\Delta d}{d_0}$$

The ratio of these two strains is called Poisson's ratio σ, or

$$\sigma = \frac{\text{Transverse strain}}{\text{Longitudinal strain}} = -\frac{\Delta d}{\Delta l}\frac{l_0}{d_0} \tag{7.1}$$

For most substances the value of σ lies between 0.25 and 0.5. As may be appreciated from a numerical example, problem 1, the change in cross-sectional area of a wire for any load within the elastic limit is negligible. Thus in the expression for Young's modulus the area A can be taken as a constant.

7.3 Bulk Modulus k

A second type of simple strain is the change in volume per unit volume when a substance is subjected to normal stresses over its whole surface. These normal stresses are most easily produced by immersing the substance in a fluid subjected to a pressure p. If under these conditions a normal inward force F acts on each element of area A of a substance of volume V to produce a decrease in volume of $-\Delta V$, then the bulk modulus k for this substance is

$$k = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{A}\cdot\frac{V}{-\Delta V} = -\frac{pV}{\Delta V}$$

where p is the increase in pressure producing the decrease in volume $-\Delta V$. The value of the bulk modulus for most common metals is of the order of 10^{12} dynes per cm^2.

7.4 Shear Modulus or Modulus of Rigidity n

This third and final type of simple strain is that in which successive layers of the material are moved or sheared by tangential surface

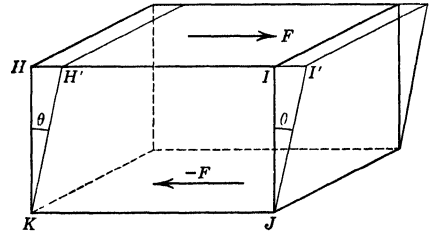


FIG. 7.3 Shearing of a rectangular block by a tangential force F.

forces. This is represented in an exaggerated manner in Fig. 7.3. To the upper surface of the rectangular block a tangential force is

applied, and an equal and opposite force $-F$ to the lower surface. As a result the upper surface is sheared relative to the lower. If the lower surface is considered fixed, then the upper edge is moved from the position HI to $H'I'$. Each layer is moved or sheared by an amount proportional to its distance from the lower fixed surface. The strain is measured by the ratio of the horizontal displacement of any layer to its distance from the fixed surface:

$$\text{Shearing strain} = \frac{HH'}{HK} = \tan \theta$$

where θ is called the angle of shear and is always very small if the elastic limit is not to be exceeded.

The shearing stress is the tangential force F divided by the area A over which it acts. Thus the modulus of rigidity n is given for Fig. 7.3 as

$$n = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{A \tan \theta}$$

For most common metals n is of the order of 10^{11} dynes per cm^2 as shown in the following table.

APPROXIMATE VALUES OF ELASTIC MODULI

Substance	Young's Modulus dynes/cm^2	Shear Modulus dynes/cm^2	Bulk Modulus dynes/cm^2	Poisson's Ratio
Aluminum	7.0×10^{11}	2.4×10^{11}	$7\ 5 \times 10^{11}$	0 34
Cast bronze	$8\ 1 \times 10^{11}$	$3\ 4 \times 10^{11}$	9.6×10^{11}	0 18
Copper	$12\ 3 \times 10^{11}$	$4\ 5 \times 10^{11}$	$13\ 1 \times 10^{11}$	0.34
Gold	8.0×10^{11}	$2\ 8 \times 10^{11}$	16.6×10^{11}	0.42
Lead	1.6×10^{11}	$0\ 54 \times 10^{11}$	$5\ 0 \times 10^{11}$	0.45
Silver	7.8×10^{11}	2.8×10^{11}	10.9×10^{11}	0.37
Steel	$20\ 6 \times 10^{11}$	$8\ 9 \times 10^{11}$	18.1×10^{11}	0.33
Tin	4.5×10^{11}	1.67×10^{11}	$5\ 1 \times 10^{11}$	0 31
Quartz fiber	5.2×10^{11}	3.0×10^{11}	$1\ 4 \times 10^{11}$	...
Glass crown	$7\ 0 \times 10^{11}$	3.0×10^{11}	5.0×10^{11}	0.24
Phosphor bronze	$12\ 0 \times 10^{11}$	4.3×10^{11}	. . . . .	0 36

7.5 Relationship of Shearing to Tensile Stresses

After considering three simple stresses we shall show how these are related in a particular case. Consider the rectangular parallelopiped *abcd*, Fig. 7.4*a*, each of whose faces at *ab* and *cd* perpendicular to the X axis have an area A_x and are subjected to a uniform tensile force F_x. Similarly the faces at *bc* and *da* perpendicular to the Y axis each have an area A_y and are subjected to a uniform tensile force F_y. If we imagine that the rectangular parallelopiped is cut along the diagonal

ca, then there is formed a prismatic section *acd*, **Fig. 7.4*b***. The forces acting on this section are F_x over the face at *cd*, F_y over the face at *ad*, a force N normal to the face at *ca*, and a surface or shearing force

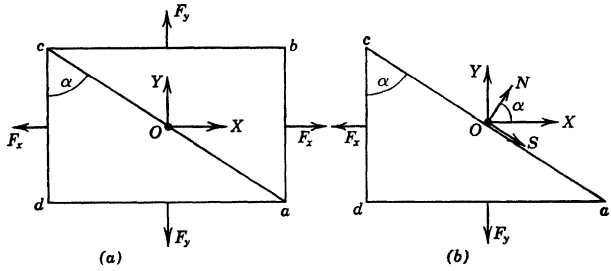


FIG. 7 4 Equilibrium of forces on a rectangular block.

S along the face at *ca*. These four forces must keep the prismatic slab in equilibrium. For this to occur, from Fig. 7.4*b*

$$N = F_x \cos \alpha + F_y \sin \alpha \tag{7.2}$$

$$S = F_x \sin \alpha - F_y \cos \alpha \tag{7.3}$$

If A is the area of the diagonal face at *ac*, then

$$A_x = A \cos \alpha \qquad \text{and} \qquad A_y = A \sin \alpha$$

Let s_x and s_y be the tensile stresses over the areas A_x and A_y respectively. Then

$$s_x = \frac{F_x}{A_x} = \frac{F_x}{A \cos \alpha} \qquad \text{and} \qquad s_y = \frac{F_y}{A_y} = \frac{F_y}{A \sin \alpha}$$

Similarly let s_n and s_s be the normal and shearing stresses respectively over the diagonal face whose area is A so that

$$s_n = \frac{N}{A} \qquad \text{and} \qquad s_s = \frac{S}{A}$$

Substituting for N and S in Eqns. 7.2 and 7.3 we obtain

$$s_n = s_x \cos^2 \alpha + s_y \sin^2 \alpha$$

$$= \frac{s_x}{2}(1 + \cos 2\alpha) + \frac{s_y}{2}(1 - \cos 2\alpha)$$

$$s_n = \frac{s_x + s_y}{2} + \frac{s_x - s_y}{2} \cos 2\alpha \tag{7.4}$$

and

$$s_s = (s_x - s_y) \sin \alpha \cos \alpha = \tfrac{1}{2}(s_x - s_y) \sin 2\alpha \qquad (7.5)$$

Both the normal and the shearing stresses vary with the angle α. If α is zero and s_x is larger than s_y, then, as may be seen from Fig. 7.4 or from Eqns. 7.4 and 7.5, the normal stress s_n has a maximum value and the shearing stress s_s a value of zero. Similarly when $\alpha = 90°$ the normal stress s_n is a minimum and the shearing stress s_s is zero. The planes for which the normal stress s_n has a maximum or minimum value are called the principal planes.

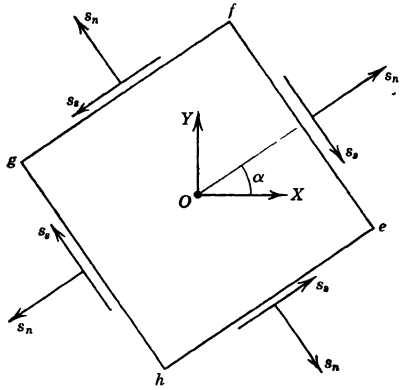


FIG. 7.5 Shearing and normal stresses on a cubical block drawn inside the rectangular block shown in Fig. 7.4.

Now imagine a cubical block *efgh*, Fig. 7.5, drawn with its center coinciding with *O*, the center of the rectangular parallelopiped shown in Fig. 7.4*a*. By analysing the other three triangles similar to the one shown in Fig. 7.4*b* it can readily be shown that the stresses over the faces *ef*, *fg*, *gh*, *he*, have the same magnitude and act in the directions shown in Fig. 7.5. For example, over the face *gh* the stresses are equal and opposite to those over the face *ef*.

7.6 Relationship between the Three Elastic Moduli, E, k, and n, and Poisson's Ratio σ

The problem is to show that for a homogeneous isotropic body

$$E = 3k(1 - 2\sigma) = 2n(1 + \sigma) = \frac{9kn}{2k + n}$$

In order to prove these relationships we shall consider that a cube made of an isotropic and homogeneous substance is acted on by forces at right angles to the faces. Let s_x, s_y, s_z be the stresses or forces per unit area acting on the faces of the cube placed as shown in Fig. 7.6. Suppose that the magnitude of each of these stresses is the same. If s_x acted alone, it would produce an extension per unit length in the

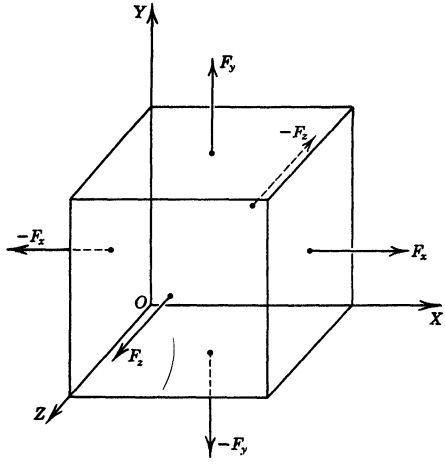


FIG. 7.6 Cubical block acted on by normal forces.

X direction of s_x/E and a contraction per unit length or negative linear strain along the Y and Z directions of $\sigma s_x/E$. Similar expressions hold for the stresses s_y and s_z acting alone. Thus, if all three stresses s act simultaneously on the cube, there will result a linear strain of extension parallel to each of the three coordinate axes of

$$\frac{s}{E}(1 - 2\sigma)$$

From the definition of the bulk modulus k these stresses would produce an increase in volume per unit volume, or volume strain, of s/k. Now it is left as an exercise to prove that a volume strain is equivalent to three mutually perpendicular linear strains, each equal to one-third of the volume strain. From this result it then follows that

$$\frac{s}{3k} = \frac{s}{E}(1 - 2\sigma)$$

or

$$E = 3k(1 - 2\sigma) \tag{7.6}$$

Let us now consider a rectangular parallelopiped on which are acting both the tensile force F_x and the compressive force F_y over the areas A_x and A_y respectively, Fig. 7.7. The corresponding stresses $s_x = F_x/A_x$ and $s_y = F_y/A_y$ are assumed to have the same magnitude. If an imaginary cube *abcd* is drawn within the rectangular parallelopiped with common centers at O and the faces inclined at 45° to the X and Y axes, then from Eqn. 7.4 the normal stress s_n is zero, since in this problem $s_x = -s_y$. From Eqn. 7.5 the shearing stress s_s is equal to s_x. This cube is therefore acted on only by shearing stresses of equal magnitude and is in a state of pure shear.

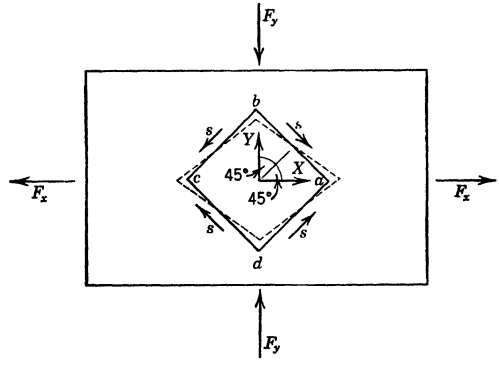


FIG. 7.7 Rectangular parallelopiped acted on by tensile and compressive forces.

Owing to these shearing stresses the cube *abcd* is deformed into the rhombus drawn in dotted lines in Fig. 7.7. Since there are no normal stresses, the sides of the rhombus are equal in length to the sides of the cube.

Now the tensile stress s_x produces an elongation per unit length or linear strain of elongation of s_x/E along the X direction and a linear strain of contraction of $\sigma s_y/E$ along the Y direction. Similarly the strain of compression s_y produces a linear strain of contraction of s_y/E along the Y axis and a linear strain of elongation of $\sigma s_y/E$ along the X axis. Thus for this case in which $s_x = -s_y = s$ the linear strain of elongation along the X axis or along the diagonal is $s/E + \sigma s/E$ and there is an equal linear strain of contraction along the Y axis.

By rotating the cube through 45° the elongation and compression

along the diagonals can be represented as in Fig. 7.8. In this the angle of shear is θ where the modulus of rigidity n is related to the shearing stress s by $n = s/\theta$. From Fig. 7.8 the linear strain of extension along the diagonal is

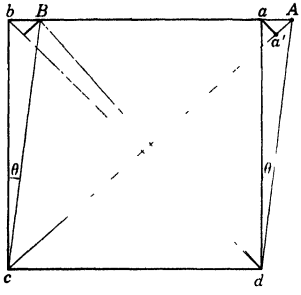


FIG. 7.8 Angle of shear θ for cube *abcd* shown in Fig. 7.7.

$$\frac{a'A}{cA} = \frac{s}{E}(1 + \sigma)$$

Now, since θ is very small, the angle aAa' is approximately 45°, and to this degree of approximation $a'A = aA/\sqrt{2}$, $cA = ca = \sqrt{2}\,ad$, and $\tan\theta = \theta = aA/ad$. Hence

$$\frac{a'A}{cA} = \frac{aA}{2ad} = \frac{\theta}{2} = \frac{s}{E}(1 + \sigma)$$

Thus, substituting s/n for θ, we have

$$\frac{s}{2n} = \frac{s}{E}(1 + \sigma)$$

or

$$E = 2n(1 + \sigma) \tag{7.7}$$

Equations 7.6 and 7.7 are two independent equations relating three of the four quantities E, n, k, σ. It is not possible to have any more independent equations between these four quantities, but it is possible to have two other equations relating three different quantities. Thus, if σ is eliminated from Eqns. 7.6 and 7.7, there results

$$E = \frac{9kn}{3k + n} \tag{7.8}$$

and, eliminating E, we have

$$\sigma = \frac{3k - 2n}{6k + 2n} \tag{7.9}$$

From Eqns. 7.6 and 7.7 it follows that

$$3k(1 - 2\sigma) = 2n(1 + \sigma)$$

and since k and n are positive quantities it follows that Poisson's ratio σ cannot be greater than one-half and cannot be less than minus one.

We have emphasized that the foregoing theory applies only to

homogeneous isotropic substances. For anisotropic substances no such simple relationships exist. It is shown in more advanced treatises that for anisotropic substances there are twenty-one elastic constants. The general expressions for stresses and strains are called dyadics or tensors, and each contains nine terms. These of course reduce to the simple expressions we have obtained for isotropic substances.

7.7 Young's Modulus as Obtained from the Bending of Beams

We have already shown how Young's modulus for a wire can be obtained by measuring the extension produced by a known load. A less obvious method for obtaining Young's modulus utilizes the deflection of a beam under a given load.

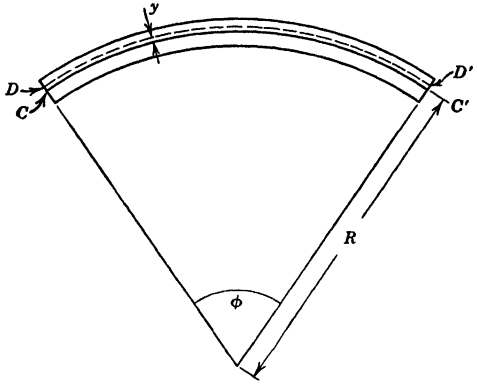


FIG. 7.9 The bending of a rod

Suppose that a rod such as a meter stick is slightly bent into a circular arc of radius R. There is a surface passing through CC' in Fig. 7.9 that does not undergo any change in length with the bending, whereas surfaces above and below CC' respectively undergo extension and contraction. The surface through CC' that remains the same length before and after bending is called the *neutral surface* or neutral plane. It will now be shown that the neutral surface passes through the center of mass of the rod.

Consider a surface DD' at a distance y above the neutral surface CC'. From Fig. 7.9 it follows that

$$CC' = R\phi \qquad \text{and} \qquad DD' = (R + y)\phi$$

so that the elongation of DD' is $y\phi$. Thus the linear strain of the

surface DD', or its elongation divided by its original length, is $y\phi/R\phi$ or y/R. At a distance y below the neutral surface there is similarly a linear strain of contraction of $-y/R$. These strains increase from zero at the neutral surface to a maximum at the surface of the rod. Corresponding to each of these strains there are stresses on the different

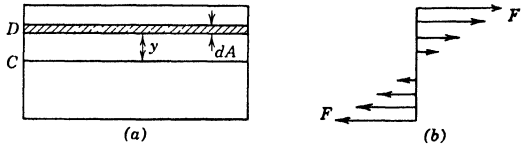


FIG. 7.10 Forces acting on a cross section of a bent rod.

layers. In order to evaluate these stresses consider a thin layer of area dA in a cross section of the rod at a distance y above the neutral surface as shown in Fig. 7.10. If F is the force acting perpendicular to the area dA, then by definition of Young's modulus

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{dA} \cdot \frac{R}{y}$$

or

$$F = \frac{Ey\,dA}{R} \tag{7.10}$$

In Fig. 7.10*b* these forces are shown in a cross section of the rod. They act in opposite directions above and below the neutral surface and produce elongations and contractions respectively. These forces are in translational equilibrium so that their vector sum is zero or

$$\Sigma F = 0$$

and from Eqn. 7.10 it follows that

$$\frac{E}{R} \sum y\,dA = 0$$

Since neither E nor R is zero, then $\Sigma y\,dA$ must be equal to zero. If the rod is of uniform density, the masses dm are proportional to the areas dA so that $\Sigma y\,dm = 0$. It thus follows from the definition of center of mass that the neutral surface from which y is measured must pass through the center of mass of the beam.

Now in order for the beam to be bent there must be external torques applied to the beam. These must be balanced by or in equilibrium with the moments of the internal stress forces. The moment of the

stress force acting on dA at a distance y from the neutral surface is

$$Fy = \frac{E}{R} y^2 \, dA$$

and the total moment of all the forces acting over the whole cross section of the rod is

$$\sum \frac{E}{R} y^2 \, dA = \frac{E}{R} \sum y^2 \, dA = \frac{E}{R} I_A$$

where

$$I_A = \Sigma y^2 \, dA$$

The quantity I_A is very similar to the corresponding moment of inertia $\Sigma y^2 \, dm$. For this reason I_A is often called the *second moment of area*. Like the moment of inertia its value depends on the axis about which it is taken.

Let us consider a rod whose second moment of area is I_A and whose value for Young's modulus is E. If an external torque L acts on this rod so as to bend it in an arc of a circle whose radius is R, then

$$L = \frac{E}{R} I_A \tag{7.11}$$

This is the fundamental equation for the bending of beams.

It is often useful to express the radius of curvature R in terms of the derivatives in an XY coordinate system. Thus, as is proved in most elementary calculus books

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \tag{7.12}$$

In all the cases that we shall consider the deflections are within the elastic limit and the slope dy/dx at any point in the beam is so small that it will be sufficiently accurate to take R as

$$R = \frac{1}{d^2y/dx^2}$$

Thus from Eqn. 7.11 the external torque L is given by

$$L = EI_A \frac{d^2y}{dx^2} \tag{7.13}$$

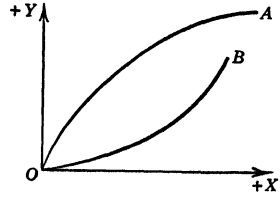


FIG. 7.11 Curves having positive and negative curvatures.

In applying this equation one has to be careful of the signs of the terms, and for this purpose the curves in Fig. 7.11 are drawn. Along OA in Fig. 7.11

the slope dy/dx of the curve is continuously decreasing so that along this curve d^2y/dx^2 is negative. Similarly along OB the slope dy/dx is continuously increasing so that along OB d^2y/dx^2 is positive. The torque L about any point is positive whenever it has a direction such as to tend to cause the rod to have a positive curvature, that is, to make d^2y/dx^2 positive. Conversely L is negative if it tends to produce a negative curvature.

7.8 The Shape of a Heavy Beam Clamped at One End and Loaded at the Other End

Let us determine the shape and depression of a uniform heavy beam clamped at one end and loaded at the free end. Suppose that the beam has a length of l and a weight of W'. A load W is placed on the free end as shown in Fig. 7.12. Consider any point x, y on the beam

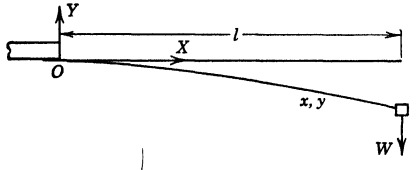


FIG. 7.12 Deflection of a horizontally clamped heavy beam.

with the X, Y axes drawn as shown in the figure. The torque exerted by the forces to the right of the point x, y is balanced by the internal torque in the beam exerted at the point x, y. From Eqn. 7.13 for the moments about the point x, y

$$EI_A \frac{d^2y}{dx^2} = -W(l - x) - \frac{W'}{2l}(l - x)^2$$

In this case d^2y/dx^2 is negative along the bent rod, and the moments that tend to bend the rod in the shape shown must be negative. The second term on the right-hand side of the equation is that due to the weight $W'(l - x)/l$ of the portion of the rod of length $(l - x)$ acting at its center of gravity a distance of $(l - x)/2$ from the point x, y. Integration gives

$$EI_A \frac{dy}{dx} = \frac{W}{2}(l - x)^2 + \frac{W'}{6l}(l - x)^3 + C_1$$

where C_1 is the constant of integration and is evaluated from the condition $dy/dx = 0$ at $x = 0$. Hence

$$C_1 = -\frac{Wl^2}{2} - \frac{W'l^2}{6}$$

Integrating a second time gives

$$EI_A y = -\frac{W}{6}(l-x)^3 - \frac{W'(l-x)^4}{24l} - \frac{Wl^2x}{2} - \frac{W'l^2x}{6} + C_2$$

where C_2, the constant of integration, is evaluated from the condition $y = 0$ at $x = 0$. Hence

$$C_2 = \frac{Wl^3}{6} + \frac{W'l^3}{24}$$

Thus the deflection y of the beam at any point x is given by

$$EI_A y = -\frac{W}{6}(l-x)^3 - \frac{W'(l-x)^4}{24l} - \frac{Wl^2x}{2} - \frac{W'l^2x}{6} + \frac{Wl^3}{6} + \frac{W'l^3}{24} \quad (7.14)$$

and the deflection y at the free end where $x = l$ is

$$y = -\frac{l^3}{EI_A}\left(\frac{W}{3} + \frac{W'}{8}\right) \quad (7.15)$$

Thus the effect of the weight W' of the beam on the deflection at the end of the beam is the same as if the beam were weightless and a weight of $3W'/8$ added to the end.

7.9 The Shape of a Heavy Beam Clamped at One End and Supported at the Other End

The problem is to determine the shape and the depression at the center of a uniform beam clamped horizontally at one end and supported at the same level at the other end. Let us suppose that a beam having a length l and a weight W is supported as shown in Fig. 7.13. Let the clamped end A be the origin of the X, Y coordinates. At A

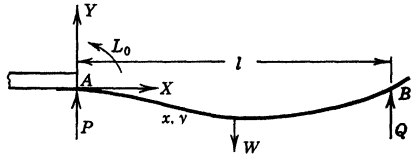


FIG. 7.13 Beam clamped at one end and supported at the other end.

there is an upward supporting force P and a torque L_0 due to the clamp; at end B there is only an upward supporting force Q. Owing to the nature of the supports the beam is not symmetrical about the mid-point. For translational equilibrium

$$P + Q - W = 0$$

The torque about A is

$$Ql + L_0 - \frac{Wl}{2} = 0$$

From Eqn. 7.13 for a point x, y we have

$$EI_A \frac{d^2y}{dx^2} = Px - L_0 - \frac{Wx^2}{2l}$$

where E is Young's modulus for the beam and I_A is the second moment of area of a cross section. The term $Wx^2/2l$ is the torque due to the weight Wx/l of the portion of the beam between A and x whose center of gravity is at $x/2$. By substitution for P and L_0 we have

$$EI_A \frac{d^2y}{dx^2} = Wx - Qx - \frac{Wl}{2} + Ql - \frac{Wx^2}{2l}$$

Integrating this equation and using the boundary conditions that, at $x = 0$, $y = 0$ and $dy/dx \neq 0$ gives

$$EI_A \frac{dy}{dx} = \frac{Wx^2}{2} - \frac{Qx^2}{2} - \frac{Wlx}{2} + Qlx - \frac{Wx^3}{6l}$$

and

$$EI_A y = \frac{Wx^3}{6} - \frac{Qx^3}{6} - \frac{Wlx^2}{4} + \frac{Qlx^2}{2} - \frac{Wx^4}{24l}$$

In order to determine Q in terms of W we must use another boundary condition, namely, $y = 0$ at $x = l$. This gives

$$Q = \frac{3W}{8}$$

Thus the deflection y at any point x is

$$y = \frac{Wx^2}{48EI_A l}(5xl - 3l^2 - 2x^2)$$

and the deflection at the mid-point $x = l/2$ is

$$y_{l/2} = -\frac{Wl^3}{192EI_A}$$

As an example let us suppose that the beam is a meter stick whose length l is 100 cm, whose weight W is 150 gm, whose width is 2.5 cm, and whose thickness

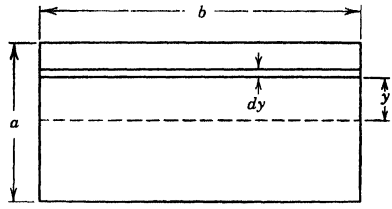


FIG. 7.14 Second moment of area of a rectangle.

is 0.5 cm. Young's modulus E for the beam is 0.5×10^{11} dynes per cm^2. If the meter stick is placed with the 2.5-cm side horizontal, then from Fig. 7.14

$$I_A = \sum y^2\,dA = \int_{-a/2}^{a/2} y^2 b\,dy$$

$$= \frac{ba^3}{12} = \frac{2.5 \times (0.5)^3}{12} = \frac{0.625}{24}\ \text{cm}^4$$

Hence

$$y_{l/2} = -\frac{150 \times 980 \times 10^6 \times 24\ \dfrac{\text{dynes cm}^3}{\text{cm}^2}}{192 \times 5 \times 10^{10} \times 0.625\ \text{dynes}}\ \text{cm}^4$$

$$= -0.588\ \text{cm}$$

or the center of the beam is depressed a distance of 0.588 cm below the horizontal owing to its own weight.

7.10 Period of Vibration of a Light Horizontal Beam Carrying a Load

The problem is to determine the period of oscillation of a light beam of length l clamped at one end and carrying a load W at the other as

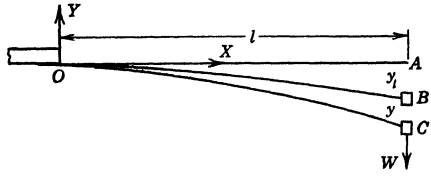


FIG. 7.15 Period of oscillation of a horizontally clamped light beam carrying a load W.

shown in Fig. 7.15. If the beam is horizontal before the load W is added, then this load causes a depression AB given by Eqn. 7.15 as

$Wl^3/3EI_A$ since the beam is assumed to be weightless or $W' = 0$. The equilibrium position of the beam with the load W on it is OB.

Suppose now that the end of the beam is pulled down from B to C, a distance y below the equilibrium position. If F is the applied force, then there is a total deflection of $(y_l + y)$ due to the total force $(W + F)$ where, by Eqn. 7.15,

$$(y_l + y) = -\frac{l^3}{EI_A}\left(\frac{W+F}{3}\right)$$

By subtraction of the value of y_l it follows that

$$y = -\frac{l^3F}{3EI_A}$$

The force F required for a displacement y below the equilibrium position is

$$F = -\frac{3EI_Ay}{l^3}$$

If the beam is released when in the displaced position, it starts upward with an acceleration y given by Newton's second law as

$$\ddot{y} = \frac{F}{M} = \frac{F}{W/g} = -\frac{3EI_Agy}{Wl^3} \tag{7.16}$$

where the mass of the body moved is W/g. The negative sign appears because the acceleration is in the opposite direction to the displacement y. The acceleration $\ddot{y}$ in Eqn. 7.16 is proportional to and in the opposite direction from the displacement y. This equation then represents simple harmonic motion, the period of which is

$$P = 2\pi\sqrt{\frac{Wl^3}{3EI_Ag}} = 2\pi\sqrt{\frac{Ml^3}{3EI_A}} \tag{7.17}$$

In this analysis the mass of the beam has been neglected. If this is to be included, then one has to take into account the fact that different portions of the beam move with different velocities. The clamped end has zero velocity whereas the free end has maximum velocity. An analysis of this problem shows that the period of oscillation of a heavy beam of weight W' clamped at one end and carrying a load W at the other end is

$$P = 2\pi\sqrt{\frac{\left(\frac{W}{g} + \frac{33W'}{140g}\right)l^3}{3EI_A}}$$

7.11 Measurement of the Modulus of Rigidity n by a Static Method

One of the simplest means of determining the modulus of rigidity is to measure the angle of torsion in a cylindrical rod produced by a known torque. Suppose that a uniform cylindrical rod is hung vertically with its upper end rigidly clamped. The lower end is subjected to a known torque, and the resulting angle of twist is measured, Fig. 7.16*a*. Each cross section of the cylindrical rod is twisted through

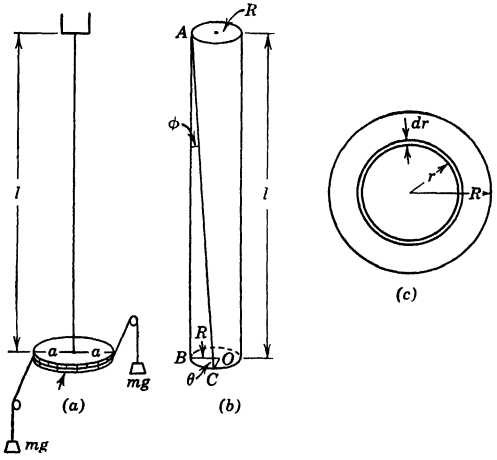


FIG. 7.16 Measurement of modulus of rigidity n.

some angle proportional to the distance of the cross section from the upper fixed end.

Suppose that the cylindrical rod has a length l and a radius R, Fig. 7.16*b*, and that a straight line AB on the unstrained rod is twisted into position AC by the torque at the lower end. The angle BAC on the rod is ϕ, and the angle through which the point B is turned is θ. From the geometry of the figure

$$l\phi = R\theta = \text{arc } BC$$

Consider now a magnified cross section of the lower end of the rod, Fig. 7.16*c*. The shear on this is measured by the angle ϕ. If F is the shearing force on an annulus of radius r and thickness dr, then the coefficient of rigidity n is

$$n = \frac{F}{2\pi r\, dr}\frac{1}{\phi}$$

or

$$F = 2\pi n\phi r\, dr$$

The moment of F about the center O is

$$Fr = 2\pi n\phi r^2\, dr$$

For this annulus

$$l\phi = r\theta$$

where θ is a constant amount over the cross section. Hence

$$Fr = \frac{2\pi n\theta}{l} r^3\, dr$$

These shearing moments act over the whole cross section, and their total sum is equal to the applied moment of force or torque L. Thus

$$\int_0^R Fr = L = \int_0^R \frac{2\pi n\theta}{l} r^3\, dr$$

or

$$L = \frac{\pi n\theta R^4}{2l} \tag{7.18}$$

The torque required to produce an angle of twist of 1 radian is often called L_0, the torsion constant, and is equal to L/θ, or

$$L_0 = \frac{\pi nR^4}{2l}$$

Thus, by measuring the quantities L_0, R, and l, the modulus of rigidity n may be obtained. In Fig. 7.16*a* the torque L is equal to $2mga$ and θ is measured on a graduated scale so that L_0 is readily obtained.

7.12 Measurement of the Modulus of Rigidity n by a Dynamic Method

Suppose that the cylindrical rod in Fig. 7.16*a* is fixed at its upper end and has a concentric circular disk attached to its lower end. If the lower end and disk are twisted through some angle θ, then the system is subjected to a torque given by Eqn. 7.18. This torque produces an angular acceleration $\ddot{\theta}$. If I is the moment of inertia of the system about an axis through the longitudinal axis of the rod, then the equation of motion of the system is

$$\frac{\pi nR^4\theta}{2l} = -I\ddot{\theta} \tag{7.19}$$

The negative sign appears because the angular acceleration is toward the origin while θ increases away from the origin. Equation 7.19 represents angular harmonic motion whose period P is

$$P = 2\pi\sqrt{\frac{2lI}{\pi nR^4}} \tag{7.20}$$

The moment of inertia I is the sum of the moments of the inertia of the disk and of the cylindrical rod about the central axis. In general the moment of inertia of the disk is so much larger than the moment of inertia of the rod that the latter may be neglected. From Eqn. 7.20 it follows that the modulus of rigidity of the material used in the rod shown in Fig. 7.16 is given by

$$n = \frac{8\pi lI}{P^2R^4}$$

By placing two similar bodies on the disk at equal distances from the axis and then determining the new period, it is readily seen that the moment of inertia of the bodies can be obtained. This is a fairly common method of obtaining the moment of inertia of unsymmetrical bodies since it is not easy to calculate them.

PROBLEMS

1. A wire 100 cm in length and 0.05 cm in radius is clamped at its upper end and has a load of 1 kg on its lower end. If Young's modulus is 6×10^{11} dynes per cm^2 and Poisson's ratio is 0.3, find the extension of the wire and the decrease in the radius due to the lateral strain.

2. Find the work done by the stretching force, and find the potential energy of the stretched wire in problem 1.

3. Show that for a homogeneous isotropic substance a volume strain is approximately three times the corresponding linear strain. As a method of proof consider a cube whose length of side is l, subjected to a uniform pressure p producing a decrease in length of the sides of Δl.

4. Show that, when a shearing stress T shears a body through an angle θ, the work done per unit volume is $\frac{1}{2}T\theta$.

5. Show that, when a uniform stress p produces a volume strain v, the work done per unit volume is $\frac{1}{2}pv$.

6. A cubical block of material is subjected to a uniform tension perpendicular to one pair of faces.

a. Show that the change in volume per unit volume is approximately $\alpha(1 - 2\sigma)$, where α is the longitudinal strain in the cube and σ is Poisson's ratio for the material.

b. Show that the fractional decrease in area of a cross section perpendicular to the tension is approximately $2\alpha\sigma$.

7. A cylindrical bar of copper 6 cm^2 in cross section and 100 cm long at 0°C is rigidly held between two end blocks. If the coefficient of linear expansion of copper is 1.7×10^{-5} per °C and the copper is heated through 50°C, find the force on the end blocks

8. A cylindrical rod of copper 0.5 cm in radius and 100 cm in length is welded at one end to a rod of steel 0.5 cm in radius and 200 cm in length. The pair of rods is hung vertically from a fixed support, and the lower end has a weight placed on it. If the lower end is extended a distance of 2.3 mm, find the load placed on the end and the extension of each rod.

9. Find the increase in density of water at 99 ft below the surface of a lake if the pressure increases by 1 atm for each 33 ft of depth. The bulk modulus of water is about 2×10^4 atm, and the normal density of water is about 62.5 lb per ft^3. (Using 1 lb of water, find its decrease in volume, its new volume and density, and then the increase in density.)

10. Show that the bulk modulus k may be expressed $k = \rho\, dp/d\rho$, where ρ is the density of the substance. If the rate of change of pressure p with depth y below the surface of a lake is given by $dp/dy = \rho g$, find the expression for the density at any depth y below the surface, assuming ρ_0 is the density at the surface where $y = 0$.

11. If the pressure p of a gas varies with volume v so that $pv =$ constant when the temperature is constant, show that the bulk modulus of a gas under these conditions is equal to its pressure.

12. When a gas is thermally insulated its pressure and volume are related by $pv^\gamma =$ constant, where γ is the ratio of the specific heat at constant pressure to that at constant volume for the gas. Show that for a thermally insulated gas or for adiabatic changes in the gas the bulk modulus is $k = \gamma p$.

13. A light rod of length l is supported at its two ends in a horizontal position, and a load W is placed on the center of the rod. If the coordinate axes have their origin at the center of the rod, show that the equation for the deflection y at any distance x is

$$y = \frac{W}{48EI_A}[6lx^2 - 4x^3 - l^3]$$

where E is Young's modulus and I_A is the second moment of area of the beam.

14. A light rod of length l is clamped at its two ends in a horizontal position, and a load W is placed on the center of the rod. If the coordinate axes have their origin at the center of the rod, show that the equation for the shape of the rod is

$$y = \frac{W}{192I_AE}[12lx^2 - 16x^3 - l^3]$$

15. A uniform beam of length l and weight W is supported at each end at the same horizontal level. Find the equation of the shape of the beam for axes whose origin is at the center of the beam.

16. A uniform beam of length l and weight W is clamped at each end at the same horizontal level. Find the equation of the shape of the beam in the region of positive x for axes whose origin is at the center of the beam.

17. A uniform beam of length l and weight W has a weight W' placed at its center and is supported at each end at the same horizontal level. Find the equation of the shape of the beam in the region of positive x for axes whose origin is at the center of the beam.

18. A uniform beam of length l and weight W has a weight W' placed at its center, and the two ends are clamped at the same horizontal level. Find the equation of the shape of the beam in the region of positive x for axes whose origin is at the center of the beam.

19. A light beam of length l is supported at each end at the same horizontal level. A load W is placed on the beam at a distance of a quarter of the length of the rod from the left-hand support. Show that the equation for the shape of the beam between O and $l/4$ is

$$y_1 = \frac{Wx}{8EI_A}\left(\frac{7}{16}l^2 - x^2\right)$$

and between $l/4$ and l is

$$y_2 = \frac{W(x-l)}{24EI_A}\left(x^2 - 2xl + \frac{l^2}{16}\right)$$

20. A light cylindrical wooden rod 0.5 in in diameter and 30 in. long is horizontally supported at its two ends so as to be used as a clothes hanger. When a suit weighing 5 lb is suspended at the midpoint, it is found that this point is depressed a distance of 0.5 in Find Young's modulus for the rod in pounds weight per square foot.

21. A cylindrical wooden rod 1 in. in diameter, 20 in. in length, and 1 lbw in weight is clamped horizontally at one end. If the free end is found to be depressed owing to its own weight by 0.03 in , find Young's modulus for the rod in pounds per square inch and the period of vibration of the rod.

22. If the rod in problem 21 has a weight of 2 lb hung on its free end, find the total depression and the period of vibration of the system.

23. A torque of 200 lbw in. applied to one end of a cylindrical rod 0.5 in. in diameter and 12 in. in length produces a twist of 1° when the other end is fixed. Find the modulus of rigidity of the rod in pounds per square inch and the work done by the torque in foot pounds weight.

24. A steel wire 1 mm in diameter and 100 cm long is suspended from a fixed support at the upper end while the lower end is rigidly attached to the center of a steel disk 10 cm in radius and 4 cm thick. If the plane of the steel disk is horizontal, find the period of the system, assuming that the density of the steel is 7.8 gm per cm^3.

25. Taking the values of Young's modulus and the shear modulus from the table for copper and steel, find the values of the bulk modulus and Poisson's ratio for these substances, assuming them to be homogeneous isotropic substances.

26. A uniform wooden rod has a mass of 200 gm, a length of 100 cm, a width of 4 cm, and a breadth of 2 cm and is clamped in a horizontal position about one end.

a. If Young's modulus for the rod is 0.5×10^{11} dynes per cm^2, find the deflection of the free end of the rod.

b. Find the period of oscillation of the rod.

c. If the rod has a maximum amplitude of oscillation of 2 cm, find the maximum kinetic energy of the rod, assuming no damping.

27. Calculate the torque exerted by the clamp on the end of the rod in Fig. 7.13.

28. Obtain the deflection y at any point x in the beam shown in Fig. 7.13 by considering the moments of the forces to the right of the mid-point rather than to the left of the mid-point as given in the text.

8 · STATICS

8.1 Introduction

Up to the present we have been concerned with the motion of objects under the action of forces. It follows from Newton's laws that when the resultant force on a body is zero its acceleration is zero The body can then either move with constant velocity or remain at rest, depending on the initial conditions. It is this state of rest with which statics deals. At first glance it might appear that statics is just a special and relatively unimportant case of dynamics. However, as we shall see, there are many problems that do not have their counterpart in dynamics.

8.2 Statics of a Particle

If a particle is to remain at rest, the resultant force on it must be zero. In Chapter 1 we showed how the resultant of a system of forces acting on a particle could be found by the parallelogram and polygon of forces and also by the resolution of forces into components. Even at the expense of repetition we shall again state these theorems as we shall be using them frequently.

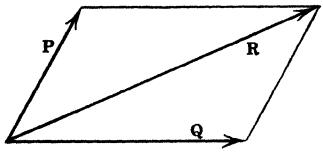


FIG. 8.1 The parallelogram of forces.

The theorem of the parallelogram of forces states that, if two forces acting at a point be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its angles, their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through this angle. In Fig. 8.1 the two forces **P** and **Q** are represented in magnitude and direction by the two sides of a parallelogram and their resultant **R** by the diagonal.

From the above theorem there follows the theorem of the triangle of forces which states that, if three forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium. Suppose that the forces **P, Q, R**

acting at O are represented in magnitude and direction by the sides **AB, BC, CA** respectively of the triangle ABC in Fig. 8.2*a*. Let us complete the parallelogram $ABCD$. Then the force represented by **AD** is **Q**, and from the theorem of the parallelogram of forces the resultant of **AD** and **AB** or of **P** and **Q** is **AC**, that is, $-\mathbf{R}$. Hence the resultant of **AB, BC,** and **CA** is equal to the resultant of **CA** and **AC** and is therefore zero, or the forces are in equilibrium.

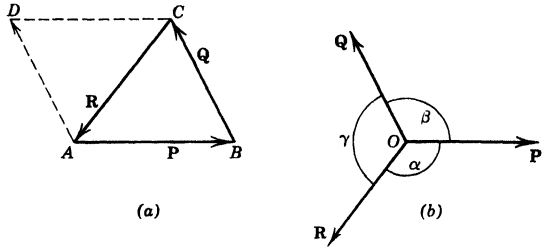


FIG. 8.2 The triangle of forces and equilibrium of three forces acting at a point.

The converse of the theorem of the triangle of forces states that, if three forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the sides of a triangle taken in order which are drawn parallel to the directions of the forces. It is this theorem that is usually proved experimentally. The theorem of the triangle of forces was first stated by Stevinus of Bruges in 1586.

There is still another theorem, called Lami's theorem, derived from the triangle of forces. This states that, if three forces acting on a particle keep it in equilibrium, each is proportional to the sine of the angle between the other two. If the three forces **P, Q, R** in Fig. 8.2*b* acting at O are in equilibrium, they can be represented by the sides AB, BC, CA respectively of the triangle ABC in Fig. 8.2*a*. From a fundamental relationship in plane trigonometry called the law of sines it follows for the triangle ABC that

$$\frac{AB}{\sin ACB} = \frac{BC}{\sin BAC} = \frac{CA}{\sin CBA}$$

Comparing Fig. 8.2*a* with Fig. 8.2*b*, it may be seen that the angle ACB is equal to the angle $(180 - \gamma)$, angle BAC is equal to $(180 - \alpha)$, and angle CBA is equal to $(180 - \beta)$. Since $\sin (180 - \alpha) = \sin \alpha$ and $\sin (180 - \gamma) = \sin \gamma$, it follows that in Fig. 8.2*b*

$$\frac{P}{\sin \gamma} = \frac{Q}{\sin \alpha} = \frac{R}{\sin \beta} \tag{8.1}$$

If more than three forces are acting at a point, then the triangle of forces becomes a polygon of forces. The theorem of the polygon of forces states that, if any number of forces, acting on a particle, can be represented in magnitude and direction by the sides of a closed polygon taken in order, then the forces are in equilibrium. In Fig. 8.3*a* the

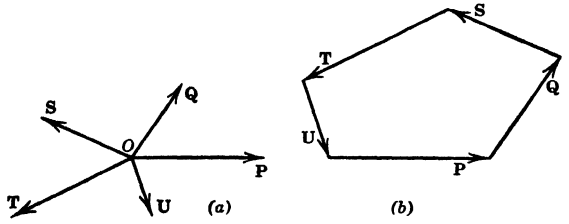


FIG. 8.3 The polygon of forces.

magnitude and direction of the forces **P**, **Q**, **S**, **T**, **U** acting at the point *O* are represented by the closed polygon shown in Fig. 8.3*b*. It is a simple extension of the theorems already given to show that, if the force polygon is closed, then the forces represented by its sides acting at some point are in equilibrium.

8.3 Resolution of Forces

Just as several forces acting at a point may be compounded into a single force, so a single force may be resolved into any number of

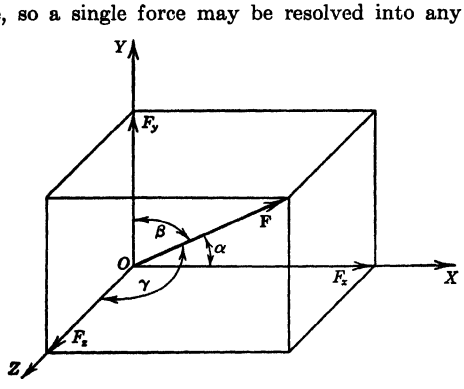


FIG. 8.4 The resolution of forces.

forces acting in different directions. A convenient direction for the resolution is frequently along the *X*, *Y*, *Z* coordinate axes. Thus in Fig. 8.4 the force **F** is resolved into the components F_x, F_y, F_z along

the X, Y, Z axes respectively. From the figure it can be seen that for the magnitude of the forces

$$F_x = F \cos \alpha \qquad F_y = F \cos \beta \qquad F_z = F \cos \gamma$$

Also

$$F^2 = F_x^2 + F_y^2 + F_z^2$$

and

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

If **i**, **j**, **k** are the unit vectors along the X, Y, Z axes respectively, then the force vector **F** may be written

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

8.4 Forces on a Simple Crane

Suppose that a crane is made up of a light wooden rod AB and a cord CB supporting a weight W as shown in Fig. 8.5*a*. Let T be the

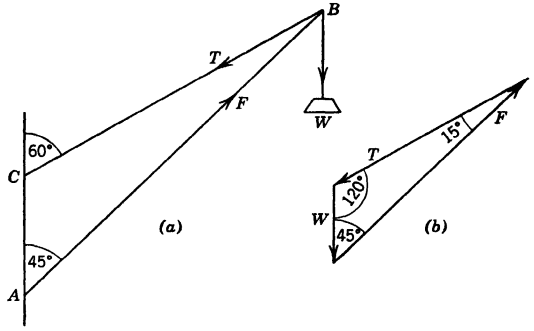


FIG. 8.5 A simple crane and its force triangle.

tension in the cord and F the thrust in the wooden rod. Since the three forces W, F, T act at the point B and are in equilibrium, they can be represented by the force triangle shown in Fig. 8.5*b*. From Lami's theorem or the law of sines:

$$\frac{W}{\sin 15^\circ} = \frac{T}{\sin 45^\circ} = \frac{F}{\sin 120^\circ}$$

Thus

$$T = W \frac{\sin 45^\circ}{\sin 15^\circ} = W \frac{0.707}{0.259} = 2.73W$$

$$F = W \frac{\sin 120^\circ}{\sin 15^\circ} = W \frac{0.866}{0.259} = 3.34W$$

The problem may also be solved by resolving the forces horizontally and vertically. Thus

$$F \cos 45° - T \cos 30° = 0$$

or

$$F = 1.22T$$

and

$$F \cos 45° - T \cos 60° - W = 0$$

so that

$$T = 2.73W \qquad \text{and} \qquad F = 3.34W$$

If the weight of the rod AB had been taken into consideration, then the force exerted by the rod would not have been along its length and this simple analysis would not apply.

8.5 Equilibrium of a Rigid Body

It is easily shown that, even though the resultant force on a body is zero, the body may still be capable of rotation. Suppose that a pair of equal and parallel forces lying in the same plane act in opposite directions not in a straight line. If such forces act on a body, then the body will rotate even though the resultant force is zero. Such a pair of equal but opposite forces is called a *couple*. We shall now prove that the moment of a couple about any point in its plane is constant. Consider the moment of the couple about the point O in the plane of the two forces F. From O drop a perpendicular onto the lines of action of the forces cutting these at A and B, Fig. 8.6. The moment of the couple about O in a counterclockwise direction is

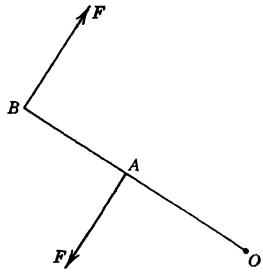


FIG. 8.6 The moment of a couple about a point.

$$(F \times AO) - (F \times BO) = F \times AB$$

Thus the moment of a couple about any point in its plane is the product of one of the forces and the perpendicular distance between the lines of action of the forces. Thus two conditions are necessary for the equilibrium of any rigid body:

a. For *translational equilibrium* the resultant force on the body must be zero:

$$\mathbf{F} = 0 \qquad \text{or} \qquad F_x = 0, \qquad F_y = 0, \qquad F_z = 0 \qquad (8.2)$$

b. For *rotational equilibrium* the sum of the moments or torques about any point must be zero:

$$\mathbf{L} = 0$$

or

$$\mathbf{r} \times \mathbf{F} = 0 \tag{8.3}$$

We are now in a position to illustrate these principles with some special problems.

8.6 The Funicular Polygon

Suppose that a light cord is attached at the two points A and B and has weights W_1, W_2, W_3, $\cdots$, W_n hung as shown in Fig. 8.7. Let the inclinations of the portions of the cord to which the weights are

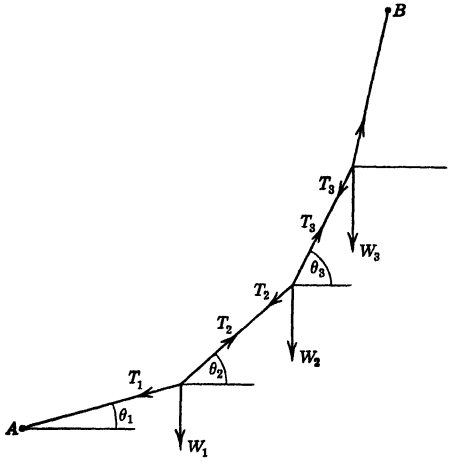


FIG. 8.7 The funicular polygon.

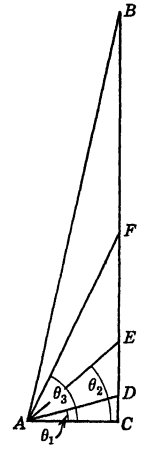


FIG. 8.8 The force triangles for the funicular polygon.

attached be θ_1, θ_2, θ_3, $\cdots$ respectively, and the tensions in these portions be T_1, T_2, T_3. Resolving the forces horizontally and vertically at the points where the weights are attached gives

$$T_2 \cos \theta_2 - T_1 \cos \theta_1 = 0$$

$$T_3 \cos \theta_3 - T_2 \cos \theta_2 = 0 \quad \text{etc.}$$

Thus

$$T_1 \cos \theta_1 = T_2 \cos \theta_2 = T_3 \cos \theta_3 = \cdots T_n \cos \theta_n = K$$

Also

$$T_2 \sin \theta_2 - T_1 \sin \theta_1 - W_1 = 0$$

$$T_3 \sin \theta_3 - T_2 \sin \theta_2 - W_2 = 0 \quad \text{etc.}$$

Substituting gives

$$T_1 = \frac{K}{\cos \theta_1}, \qquad T_2 = \frac{K}{\cos \theta_2}, \cdots, \qquad T_n = \frac{K}{\cos \theta_n}$$

It follows that

$$\tan \theta_2 = \tan \theta_1 + \frac{W_1}{K}$$

$$\tan \theta_3 = \tan \theta_2 + \frac{W_2}{K} = \tan \theta_1 + \frac{(W_1 + W_2)}{K} \tag{8.4}$$

If the weights $W_1, W_2, \cdots, W_n$ are all equal to W, then

$$\tan \theta_n = \tan \theta_1 + (n - 1)\frac{W}{K}$$

and the tangents of the angles of inclination of the cord increase in arithmetical progression.

We shall now give a graphical construction for the forces in the funicular polygon. From a horizontal line AC draw the angles θ_1, θ_2, θ_3, $\cdots$, θ_n given in the funicular polygon in Fig. 8.7. Let a vertical line through C intercept the angles at $D, E, F, \cdots$ as shown in Fig. 8.8. From Eqn. 8.4 we have

$$\frac{W_1}{K} = \tan \theta_2 - \tan \theta_1 = \frac{EC}{AC} - \frac{DC}{AC} = \frac{ED}{AC}$$

$$\frac{W_2}{K} = \tan \theta_3 - \tan \theta_2 = \frac{FC}{AC} - \frac{EC}{AC} = \frac{FE}{AC}$$

Thus

$$\frac{W_2}{W_1} = \frac{FE}{ED} \quad \text{etc.}$$

or the ratio of the weights is given by the ratio of the corresponding intercepts.

We see that the triangle AED is the force triangle for the point at which the weight W_1 is attached and AFE is the force triangle for the point where W_2 is attached. On the same scale that represents T_1 by AD, the force T_2 is represented by AE, W_1 by ED, T_3 by AF, and W_2 by FE.

8.7 A Bridge Framework

As a further example of graphic construction let us consider the following framework shown in Fig. 8.9. The framework is made up of light rods freely jointed and supported by pillars at A and E. Loads are placed at B, C, and D as shown in the figure. The problem

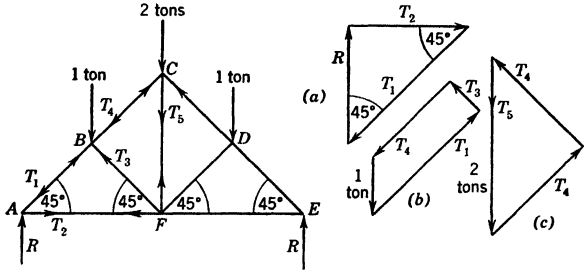


FIG. 8.9 A bridge framework and the equilibrium of the forces.

is to determine the forces in the various members of the framework. For the vertical forces, from the symmetry of the figure, we see that

$$2R = 4 \text{ tons}$$

$$R = 2 \text{ tons}$$

The force triangle for the point A is shown in Fig. 8.9*a*. By accurate measurement or calculation

$$T_2 = 2 \text{ tons} \qquad T_1 = 2.828 \text{ tons}$$

and their directions are as shown. The force polygon for the point B is shown in Fig. 8.9*b*. From the diagram

$$T_3 = 0.707 \text{ ton}$$

and

$$T_4 = 2.828 - 0.707 = 2.121 \text{ tons}$$

The force diagram for the point C is shown in Fig. 8.9*c*. By calculation or measurement

$$T_5 = 1 \text{ ton}$$

The forces T_2 and T_5 are tensions, and T_1, T_3, and T_4 represent compressions in the members. Thus such members as AF, FE, and CF could be replaced by ropes. Notice that the arrows give forces exerted by the members on the particular points considered, and the forces exerted on the members are equal and opposite to these.

8.8 The Suspension Bridge

We may consider a suspension bridge as a cable suspended between two pillars of the same height. The cable is loaded so that the weight on each element of it is proportional to the horizontal projection of that element. This is equivalent to neglecting the weight of the cable in comparison to the weight of the roadway supported by the bridge. In order to find the equation for the shape of the cable we shall draw vertical and horizontal lines through the lowest point O. These are

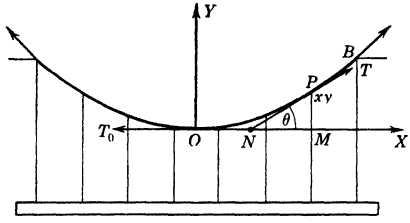


FIG. 8.10 The suspension bridge.

called the X and Y axes respectively in Fig. 8.10. Consider the equilibrium of a portion OP of the cable. The weight of this portion is proportional to the horizontal line OM, and this force acts at the center of gravity or at the middle point N of OM. The tension T in the cord at P makes an angle θ with the horizontal. If T_0 is the tension in the horizontal portion at O, then by resolving the tension at P we have

$$T \cos \theta = T_0$$

$$T \sin \theta = (\text{Weight of } OM) = wx$$

where w is the weight per unit length of the flooring and x is the horizontal coordinate of P. Then

$$\tan \theta = \frac{wx}{T_0} = \frac{dy}{dx}$$

where y is the vertical coordinate of P. By integration

$$y = \frac{wx^2}{2T_0} + C$$

Since $y = 0$ when $x = 0$, then $C = 0$, or

$$y = \frac{w}{2T_0} x^2 \tag{8.5}$$

Since w and T_0 are constants for a given bridge, it follows from Eqn. 8.5 that the cable has the shape of a parabola.

8.9 The Catenary

The problem here is to determine the shape assumed by a heavy flexible cable suspended between two points at the same horizontal level. Suppose that a uniform, inextensible, flexible cable is hung between two points A and B as shown in Fig. 8.11. Consider a point P

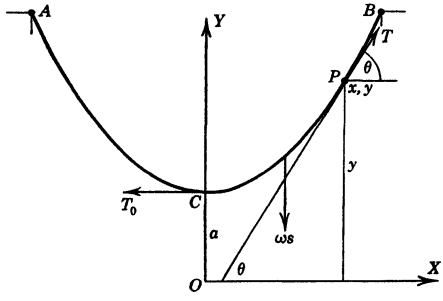


FIG. 8.11 The catenary.

of the cable where the tension T makes an angle θ with the horizontal. Let the arc CP of the cable have a length s and a weight ws where w is the weight per unit length of the cable. If T_0 is the horizontal tension at the lowest point C, then by resolution

$$T \cos \theta = T_0$$
$$T \sin \theta = ws$$

For convenience let $T_0 = wa$. Then

$$\tan \theta = \frac{s}{a} = \frac{dy}{dx} \tag{8.6}$$

where x, y are the coordinates of the point P whose distance along the arc is s. Now

$$(ds)^2 = (dx)^2 + (dy)^2$$

so that

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{s^2 + a^2}{a^2}$$

Thus

$$\frac{ds}{\sqrt{s^2 + a^2}} = \frac{dx}{a}$$

is the differential equation of the cable. By integration

$$\frac{x}{a} = \log_e (s + \sqrt{s^2 + a^2}) + C$$

where C, the constant of integration, is determined from the boundary conditions: at $x = 0$, $s = 0$. Hence

$$C = -\log_e a$$

or

$$x = a \log_e \left(\frac{s + \sqrt{s^2 + a^2}}{a} \right) = a \sinh^{-1} \frac{s}{a}$$

Hence

$$s = a \sinh \frac{x}{a}$$

From Eqn. 8.6

$$\frac{s}{a} = \frac{dy}{dx} = \sinh \frac{x}{a}$$

Hence

$$y = a \cosh \frac{x}{a} + C_1$$

where C_1 is determined from the boundary conditions. Suppose that the origin of the axes is placed at a distance a equal to CO below the lowest point. Then at $x = 0$, $y = a$. Now

$$\cosh \frac{x}{a} = \frac{e^{x/a} + e^{-x/a}}{2}$$

so that

$$C_1 = 0$$

The equation of the curve of the suspended cable is

$$y = a \cosh \frac{x}{a} \tag{8.7}$$

This curve is called a *catenary*. In problem 14 at the end of this chapter you are to show that $T = wy$ so that it follows that the tension at any point x on the catenary is $T = wa \cosh (x/a)$.

The quantity a, which defines the size of the catenary, is called its parameter and is the distance of the vertex above the origin. The horizontal line through the origin or OX is the directrix of the catenary. From the original defining equation for a, namely $a = T_0/w$, it follows that this parameter depends on how tightly the cable is stretched

and on the linear weight of the cable. If this parameter is large, then the catenary approaches a parabola. Suppose that the X axis is taken through the lowest point C of the catenary. Then from Eqn. 8.7 the equation of the catenary is

$$y = y' + a = a \cosh \frac{x}{a} = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

$$= \frac{a}{2}\left(2 + \frac{2x^2}{2a^2} + \frac{2x^4}{a^4 4!} + \cdots\right)$$

$$= a + \frac{x^2}{2a} + \frac{x^4}{24a^3} + \cdots$$

or

$$2ay' = x^2 + \frac{x^4}{12a^2} + \cdots$$

If a is made very large, that is, T_0/w is large, then approximately

$$x^2 = 2ay'$$

which is the equation of a parabola. Notice that y' is the sag of the cable.

8.10 Heavy Flexible Cable Suspended from Two Different Levels

Suppose that the heavy cable is suspended between the two points O and A in Fig. 8.12. Let O be the origin of the X, Y coordinate axes.

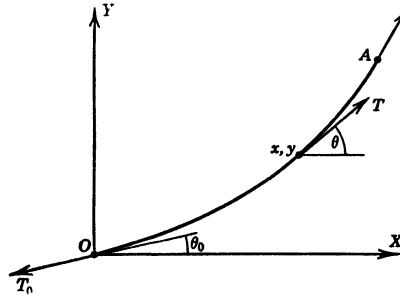


FIG. 8.12 A heavy cable suspended from points A and O.

The tension T_0 at the origin O makes an angle θ_0 with the horizontal, and the tension T at any point x, y has a corresponding angle of θ. If w is the weight per unit length of the cable, then the weight of a length ds is

$$w\,ds = w\sqrt{(dx)^2 + (dy)^2} = w\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

and the weight of the cable between the origin and the point x, y is

$$w\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Let us resolve the forces along the horizontal and vertical directions. Then

$$T\cos\theta - T_0\cos\theta_0 = 0$$

$$T\sin\theta - T_0\sin\theta_0 - w\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 0$$

Suppose that the horizontal tension is again equal to wa or a length a of the cable. Then

$$T\cos\theta = T_0\cos\theta_0 = wa$$

From this equation it follows that

$$T\sin\theta = wa\tan\theta = wa\frac{dy}{dx}$$

Hence

$$wa\frac{dy}{dx} - T_0\sin\theta_0 - w\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 0$$

By differentiation

$$wa\frac{d^2y}{dx^2} - w\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0$$

For convenience let us introduce the new variable $u = dy/dx$. Then

$$a\frac{du}{dx} = \sqrt{1 + u^2}$$

and

$$a\frac{du}{\sqrt{1 + u^2}} = dx$$

By integration we have

$$\log_e\left(u + \sqrt{1 + u^2}\right) = \frac{x}{a} + C_1$$

or

$$\sinh^{-1} u = \frac{x}{a} + C_1$$

where C_1 is the constant of integration. Thus

$$u = \frac{dy}{dx} = \sinh\left(\frac{x}{a} + C_1\right) = \tan\theta \tag{8.8}$$

At $x = 0$ the value of dy/dx is $\tan\theta_0$ so that $\sinh C_1 = \tan\theta_0$ or is equal to the tangent of the angle of slope at the origin. Integrating again gives

$$y = a\cosh\left(\frac{x}{a} + C_1\right) + C_2$$

The boundary conditions are: at $x = 0$, $y = 0$, so that

$$C_2 = -a\cosh C_1$$

Hence the value of y at any point x along the cable is given by

$$y = a\left[\cosh\left(\frac{x}{a} + C_1\right) - \cosh C_1\right] \tag{8.9}$$

In order to obtain the relationship between s and x we shall make use of the equation

$$(ds)^2 = (dx)^2 + (dy)^2$$

or

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \sinh^2\left(\frac{x}{a} + C_1\right)}$$

Now from the definitions of sinh and cosh it follows that

$$1 + \sinh^2\alpha = \cosh^2\alpha$$

Hence

$$\frac{ds}{dx} = \cosh\left(\frac{x}{a} + C_1\right)$$

By integration

$$s = a\sinh\left(\frac{x}{a} + C_1\right) + C_3$$

Since $s = 0$ at $x = 0$, it follows that $C_3 = -a\sinh C_1$. Hence

$$s = a\sinh\left(\frac{x}{a} + C_1\right) - a\sinh C_1$$

The tension T at any point is given by

$$T \cos \theta = wa$$

or

$$T = \frac{wa}{\cos \theta} = wa \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} = wa \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$T = wa \cosh \left(\frac{x}{a} + C_1\right) \tag{8.10}$$

In practice most cables are not perfectly flexible so that they do not exactly follow the equations given above. However, to take into account the lack of flexibility would be almost impossible mathematically. In order to illustrate these equations we shall take a practical example.

Example. An electric cable considered perfectly flexible and weighing 0.2 lbw per ft is suspended between two points whose distance is 1000 ft horizontally and 100 ft vertically. Find the tension at the upper and lower supports if it is observed

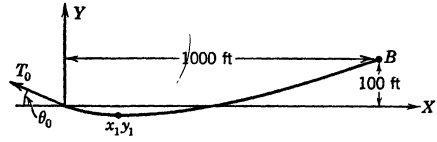


FIG. 8.13 Example of a heavy cable suspended at different levels.

that the cable sags below the lower support by 10 ft. Let the X and Y axes be taken through the lower support, and let x_1, y_1 be the coordinates of the lowest point of the cable as shown in Fig. 8.13. At this point $dy/dx = 0$. Thus from Eqn. 8.8

$$\sinh \left(\frac{x_1}{a} + C_1\right) = 0$$

and therefore

$$\frac{x_1}{a} + C_1 = 0$$

Also, since $y_1 = -10$, then from Eqn. 8.9

$$-10 = a \left[\cosh \left(\frac{x_1}{a} + C_1\right) - \cosh C_1\right]$$

or

$$-10 = a(1 - \cosh C_1)$$

Since if $\sinh x = 0$ then $\cosh x = 1$, this follows from the definitions given in the footnote.* Thus

$$\cosh C_1 = 1 + \frac{10}{a} = \frac{a + 10}{a}$$

Now for the upper support $x = 1000$, $y = 100$. Hence from Eqn. 8.9

$$100 = a\left[\cosh\left(\frac{1000}{a} + C_1\right) - \cosh C_1\right]$$

$$= a\left(\cosh\frac{1000}{a}\cosh C_1 + \sinh\frac{1000}{a}\sinh C_1 - \cosh C_1\right)$$

Since $\tan\theta_0$ is negative then from Eqn. 8.8 $\sinh C_1$ is negative, hence

$$\frac{100}{a} = \left(\frac{a + 10}{a}\right)\cosh\frac{1000}{a} - \left[\sqrt{\left(\frac{a + 10}{a}\right)^2 - 1}\right]\sinh\frac{1000}{a} - \left(\frac{a + 10}{a}\right)$$

$$110 + a = (a + 10)\cosh\frac{1000}{a} - \sqrt{100 + 20a}\sinh\frac{1000}{a}$$

This is an equation where a must be found by brute force. After several trials it was found that $a = 2690$ satisfies the equation From this value

$$\cosh C_1 = \tfrac{2700}{2690} = 1.00372$$

and from the above C_1 is negative so that

$$C_1 = -0.0861$$

The tension T at any point x is given by Eqn. 8.10. At the lower support $x = 0$. Hence

$$T_0 = wa\cosh C_1 = 0.2 \times 2690 \times 1.00372 = 540 \text{ lbw.}$$

At the upper support $x = 1000$ ft. Hence

$$T_{(x=1000)} = 538\cosh\left(\tfrac{1000}{2690} - 0.0861\right) = 538\cosh 0.2857$$

$$= 538 \times 1.0411 = 560 \text{ lbw.}$$

8.11 Tension on a Cord Wrapped around a Rough Post

Most of us are familiar with the fact that when a rope is wrapped around a rough post a small force on one end can balance a large force on the other. A capstan pulling a ship into a dock is an example

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* $$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Also

$$\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$$

and

$$\cosh^2 x - \sinh^2 x = 1$$

of this. To analyze this problem let us consider an element ds of a *light* rope resting on a rough curved surface. This element ds is pulled by tensions $(T + dT)$ upward and by the tension T downward so that the rope is on the point of slipping upward, as shown in Fig. 8.14.

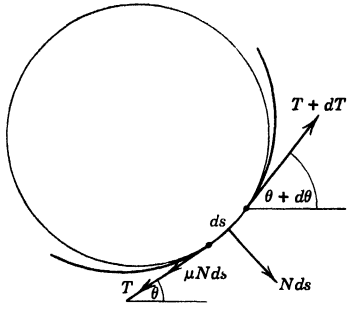


FIG. 8.14 Friction on a cord wound around a rough post.

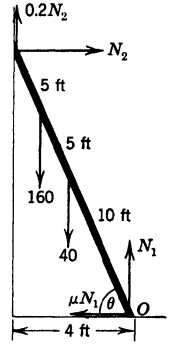


FIG. 8.15 Equilibrium of a ladder resting against a rough wall.

The force of friction on the rope then acts downward. Its value is equal to the product of the coefficient of friction μ and the normal reaction acting on the element ds. If N is the normal reaction per unit length, then the normal reaction on ds is $N\,ds$ in a direction perpendicular to the tangential force T. The force of friction on ds is $\mu N\,ds$ acting in the direction of T. Resolving the forces along the tangent and normal we have, when ds is in equilibrium,

$$(T + dT)\cos d\theta - T - \mu N\,ds = 0$$

$$(T + dT)\sin d\theta - N\,ds = 0$$

Since $d\theta$ is very small, $\cos d\theta$ may be set equal to unity and $\sin d\theta$ equal to $d\theta$ in radian measure. Hence

$$\frac{dT}{ds} = \mu N \qquad \text{and} \qquad T\frac{d\theta}{ds} = N$$

Eliminating N between these equations gives

$$\frac{dT}{ds} = \mu T\frac{d\theta}{ds}$$

or

$$\frac{dT}{T} = \mu \, d\theta$$

By integration

$$\log_e T = \mu\theta + C$$

If $\theta = 0$ at a point where $T = T_0$, then the constant of integration C is $\log_e T_0$ and

$$\log_e \frac{T}{T_0} = \mu\theta$$

or

$$T = T_0 e^{\mu\theta} \tag{8.11}$$

is the equation for the tension in the rope at an angle θ if the tension at the zero angle is T_0.

As an illustration of how the tension increases with the angle consider a rope wrapped one complete turn round a post for which $\mu = 0.6$. For one complete turn $\theta = 2\pi$ radians and

$$\frac{T}{T_0} = e^{1.2\pi} = (2.718)^{3.77} = 43.4 \quad \text{approximately}$$

If the rope is wrapped twice around the post, then the ratio of the tensions is

$$\frac{T}{T_0} = e^{2.4\pi} = (2.718)^{7.54} = 1900 \quad \text{approximately}$$

Thus it can be seen how a heavy load can be held by a comparatively small force when the rope is wrapped several times around a post.

In the foregoing examples we have used only the first part of Eqn. 8.2 involving translational equilibrium. We shall now discuss some examples of the equilibrium of rigid bodies where it is necessary to take into consideration both conditions of equilibrium.

8.12 Equilibrium of a Ladder

Suppose that a ladder whose height is 20 ft and whose weight is 40 lb rests against a vertical wall with the lower end 4 ft from the foot of the wall. If the coefficient of friction between the ladder and the wall is 0.2, the problem is to find the minimum coefficient of friction between the ground and the ladder that will prevent slipping when a man weighing 160 lb stands on the ladder 5 ft from the top.

Figure 8.15 shows the ladder together with the forces acting on it.

Because the ladder tends to slip downward, the forces of friction at the ground and wall are in such a direction as to prevent this. If the normal reaction of the ground on the ladder is N_1, then the force of friction along the ground is μN_1 where μ is the coefficient of friction. Similarly if N_2 is the normal reaction at the wall, then $0.2N_2$ is the force of friction on the ladder at the wall. The problem is to find the smallest value of μ that will prevent slipping, and to do this we resolve the forces into horizontal and vertical directions.

Resolving forces horizontally:

$$N_2 - \mu N_1 = 0$$

Resolving forces vertically:

$$0.2N_2 + N_1 - 200 = 0$$

Taking moments about O:

$$(40 \times 10 \cos\theta) + (160 \times 15 \cos\theta) - (N_2 \times 20 \sin\theta) - (0.2N_2 \times 20 \cos\theta) = 0$$

or

$$400 + 2400 - 20N_2 \tan\theta - 4N_2 = 0$$

From the figure, $\cos\theta = \frac{4}{20}$ and hence $\tan\theta = 4.90$. Thus

$$2800 = 98N_2 + 4N_2$$

or

$$N_2 = 27.5 \text{ lbw}$$

$$N_1 = 200 - 0.2N_2 = 200 - 5.5 = 194.5 \text{ lbw}$$

and

$$\mu = \frac{N_2}{N_1} = \frac{27.5}{194.5} = 0.141$$

In such problems the point about which the moments are taken may be chosen for convenience since the resultant moment about any point is zero.

8.13 Reactions at Smooth Joints

In engineering many of the members of a structure are made free to move relative to one another. This hinging of one member to another may be attained by means of a ball and socket joint, a peg fitting in a round hole, or an ordinary door hinge. In any case, if the hinge can be considered to be smooth, then the action of the hinge consists of a single force. Since the direction and magnitude of this force is usually unknown, it is generally convenient to represent the

force by two components at right angles to each other. If the weight of the pin of the hinge is negligible, then the force of the hinge on one of the bodies attached to the hinge is equal and opposite to the force on the other body. Thus in solving problems on jointed structures we shall represent the force on one member by two components at right angles and on the other member by two equal and opposite components. If the joint is not smooth, then there is a couple due to friction in addition to the force. In this discussion we shall assume that all joints are smooth.

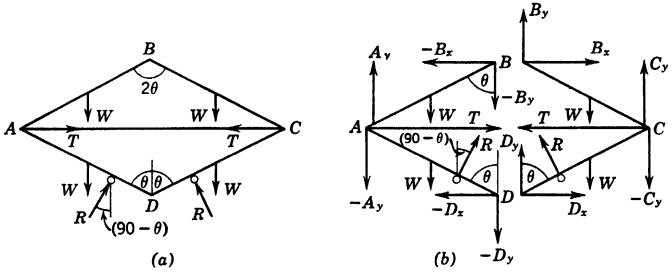


FIG. 8.16 A jointed rhombus resting on two pegs and held by a horizontal cord

As an example let us consider a rhombus, consisting of four equal rods of length l and weight W, freely jointed at the end of each rod and placed symmetrically on two smooth horizontal pegs a distance a apart. Equilibrium is maintained by a horizontal cord joining two opposite corners of the rhombus. The problem is to find the reactions at the joints and the tension in the cord.

Figure 8.16*a* shows a rhombus $ABCD$ with two of its angles equal to 2θ. Let the reaction at the pegs be R and the tension in the cord be T. The figure alongside, Fig. 8.16*b*, shows the members separated into left- and right-handed halves. This avoids difficulties with the horizontal components at the joints A and C. By resolving the forces vertically, we have for the system as a whole:

$$2R \cos (90 - \theta) = 4W$$

or

$$R = \frac{2W}{\sin \theta}$$

By resolving the forces vertically for the left section, DAB, we have

$$R \sin \theta - 2W - D_y - B_y = 0$$

Similarly for DCB

$$R \sin \theta - 2W + D_y + B_y = 0$$

From these it follows that

$$D_y = 0 \qquad B_y = 0 \qquad R = \frac{2W}{\sin \theta}$$

Taking moments about A for the rod BA, we have

$$B_x l \cos \theta = \frac{W}{2} l \sin \theta$$

or

$$B_x = \frac{W}{2} \tan \theta$$

Taking moments about A for the rod DA,

$$D_x l \cos \theta + \frac{W}{2} l \sin \theta - R\left(l - \frac{a}{2 \sin \theta}\right) = 0$$

$$D_x = -\frac{W}{2} \tan \theta + \frac{R}{\cos \theta}\left(1 - \frac{a}{2l \sin \theta}\right)$$

$$= -\frac{W}{2} \tan \theta + \frac{2W}{\sin \theta \cos \theta} - \frac{Wa}{l \sin^2 \theta \cos \theta}$$

By resolving the forces horizontally on the left section DAB, we have

$$T = B_x + D_x - R \cos \theta$$

$$= \frac{2W}{\sin \theta \cos \theta} - \frac{Wa}{l \sin^2 \theta \cos \theta} - \frac{2W \cos \theta}{\sin \theta}$$

$$= 2W \tan \theta - \frac{Wa \tan \theta}{l \sin^3 \theta}$$

Owing to the fact that the rods have weight and that there is a tension in the cord, these forces at the joints do not act along the rods.

8.14 The Principle of Virtual Work

In 1717 John Bernoulli formulated a principle associating work with the conditions of equilibrium of a body. For a body in equilibrium the components of any resultant force acting on the body in any direction must each be equal to zero or

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$

Suppose that such a body in equilibrium is considered to undergo a small displacement. This displacement need not be an actual one but may be a hypothetical or virtual one. It, of course, must be consistent with the geometrical conditions of the body. If the components of the virtual displacement are dx, dy, dz, then the virtual work done δW is

$$\delta W = \Sigma F_x\, dx + \Sigma F_y\, dy + \Sigma F_z\, dz$$

and according to the above equations must be zero. This principle of *virtual work* may be stated as follows: If a system of forces acting on a body is in equilibrium and if the body suffers a small displacement consistent with its geometrical conditions, then the total virtual work done by the forces is zero. Conversely, if the virtual work is zero the forces are in equilibrium. The usefulness of this principle will be shown by applying it to some of the examples already solved by the ordinary conditions of equilibrium.

Application of the Principle of Virtual Work to the Ladder Problem. Consider the ladder shown in Fig. 8.15 as undergoing a virtual displacement in such a manner that the angle θ increases by $d\theta$. In other words the ladder is imagined to move closer to the wall. The force of friction $0.2N_2$ at the wall is at a height of 20 sin θ feet above the ground, and when θ increases by $d\theta$ this distance increases by $d(20 \sin \theta)$ or $20 \cos \theta\, d\theta$. This is the virtual displacement of this frictional force, and the virtual work done by this force on the ladder is $0.2N_2 \times 20 \cos \theta\, d\theta$. Consider next the normal reaction force N_2. This force is perpendicular to its virtual displacement, and hence no work is done by this force. Similarly no virtual work is done by the normal force N_1 at the ground during the virtual displacement.

The algebraic sum of the virtual work done by all the forces is

$$0.2N_2\, d(20 \sin \theta) - 160\, d(15 \sin \theta) - 40\, d(10 \sin \theta) - \mu N_1\, d(20 \cos \theta)$$

and by the principle of virtual work this is equal to zero. Hence

$$(4N_2 \cos \theta - 2400 \cos \theta - 400 \cos \theta + 20\mu N_1 \sin \theta)\, d\theta = 0$$

Since $d\theta$ is arbitrary we have

$$4N_2 \cos \theta - 2800 \cos \theta + 20\mu N_1 \sin \theta = 0$$

This is the equation obtained by taking moments about O, the foot of the ladder. It contains the three unknowns μ, N_1, N_2. In order to determine these we must have two more independent equations. These can be obtained by resolving the forces horizontally and verti-

cally as was done previously. It should be noted that it is the length of the ladder that remains constant during the virtual displacement and not the initial distance of 4 ft between the foot of the ladder and the ground.

Application of the Principle of Virtual Work to a System of Jointed Rods. Let us next consider the rhombus $ABCD$ in Fig. 8.16. Suppose that the virtual displacement consists in changing the angle θ of the rhombus by $d\theta$. This flattens the rhombus and increases the diagonal AC. The distance a between the pegs and the length l of the sides of the rhombus remain constant. Since the system is symmetrical about the vertical diagonal BD, this diagonal remains vertical after the virtual displacement.

The height of the center of gravity of the rod AD above the fixed horizontal line of the pegs is $(l/2 - a/2 \sin \theta) \cos \theta$, and the virtual displacement of this point is

$$-\frac{l}{2} \sin \theta \, d\theta + \frac{a}{2} d\theta + \frac{a \cos^2 \theta \, d\theta}{2 \sin^2 \theta}$$

Similarly the center of gravity of AB undergoes a virtual displacement of

$$-\frac{3l}{2} \sin \theta \, d\theta + \frac{a}{2} d\theta + \frac{a \cos^2 \theta \, d\theta}{2 \sin^2 \theta}$$

The virtual displacement of A, the point of application of the tension force T, is

$$d\left(l \sin \theta - \frac{a}{2}\right) = l \cos \theta \, d\theta$$

The total virtual work due to the virtual displacement $d\theta$ is

$$2W\left(-\frac{l}{2} \sin \theta + \frac{a}{2} + \frac{a \cos^2 \theta}{2 \sin^2 \theta} - \frac{3l}{2} \sin \theta + \frac{a}{2} + \frac{a \cos^2 \theta}{2 \sin^2 \theta}\right) d\theta + 2Tl \cos \theta \, d\theta = 0$$

Since $d\theta$ is arbitrary, it follows that

$$Tl \cos \theta = W\left(2l \sin \theta - a - \frac{a \cos^2 \theta}{\sin^2 \theta}\right)$$

$$= W\left(2l \sin \theta - \frac{a}{\sin^2 \theta}\right)$$

or

$$T = W \tan \theta \left(2 - \frac{a}{l \sin^3 \theta}\right)$$

as given by the previous analysis.

8.15 Equilibrium and Potential Energy

From the principle of virtual work the condition for equilibrium is obtained by setting the virtual work equal to zero. This may be expressed in symbols as

$$\Sigma F_x\,dx + \Sigma F_y\,dy + \Sigma F_z\,dz = 0$$

where dx, dy, dz are the components of the virtual displacement along the X, Y, Z axes.

Let us now assume that all the forces acting on the system are conservative. From the discussion in Chapter 3 it follows that these forces are derivable from a potential energy function, where

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

and

$$\Sigma F_x\,dx + \Sigma F_y\,dy + \Sigma F_z\,dz = -dV$$

Thus the condition for equilibrium for a system of conservative forces may be written

$$dV = 0$$

that is, the potential energy of a system of conservative forces in equilibrium has either a maximum or a minimum value.

Example of equilibrium and potential energy. Let us again consider the equilibrium of the rhombus given in Fig. 8.16. Suppose that the potential energy is considered as zero at the level of the two horizontal fixed pegs. The potential energy of the rods AD and DC is

$$2W\left(\frac{l}{2} - \frac{a}{2\sin\theta}\right)\cos\theta$$

The potential energy of the cord is given by the work done against the tension in constructing the rhombus from four horizontal rods lying on the two pegs where $\theta = 90°$ to the position shown in Fig. 8.16*a*. Thus the potential energy of the cord is

$$-\int_{\pi/2}^{\theta} Tl\cos\theta\,d\theta$$

The total potential energy V of the system is

$$V = 2W\left(\frac{l}{2} - \frac{a}{2\sin\theta}\right)\cos\theta + 2W\left(\frac{3l}{2} - \frac{a}{2\sin\theta}\right) + 2\int_{\pi/2}^{\theta} Tl\cos\theta\,d\theta$$

and

$$\frac{dV}{d\theta} = -4Wl\sin\theta + 2Wa + 2\,\frac{Wa\cos^2\theta}{\sin^2\theta} + 2Tl\cos\theta$$

For equilibrium $dV/d\theta = 0$, or

$$Tl \cos \theta = W \left(2l \sin \theta - a - \frac{a \cos^2 \theta}{\sin^2 \theta} \right)$$

This is the same equation as that obtained by applying the virtual work principle. It gives for the tension in the cord

$$T = W \tan \theta \left(2 - \frac{a}{l \sin^3 \theta} \right)$$

In the ladder example there are frictional forces and hence the system is not a conservative one, so that the equilibrium condition cannot be found by the principle given in this section.

8.16 Stable and Unstable Equilibrium

In most practical cases one is interested not only in the equilibrium condition of a given system but also in seeing that this equilibrium is stable. A system in equilibrium is stable if, for a small displacement from the equilibrium position, forces are brought into play that tend to return the system to its former equilibrium position. Similarly a system is unstable if the forces resulting from a small displacement tend to move the system away from its former equilibrium position. Finally a system is in neutral equilibrium if no forces result from a small displacement that tend either to return it to or move it away from its former position. In other words it is also in equilibrium in the displaced position. A ball at the bottom of a valley is in stable equilibrium, a ball at the top of a hill is in unstable equilibrium, and a ball on a horizontal surface is in neutral equilibrium.

These conditions may be readily stated mathematically for a conservative system in equilibrium. For such a system the potential energy V is a maximum or a minimum if $\Delta V = 0$ for a very small displacement. The ball at the bottom of a valley has minimum potential energy. A small displacement increases the potential energy, and the conservative force of gravity tends to return the ball to the bottom of the hill. The potential energy can be expressed in terms of some parameter such as an angle θ or in this case the vertical height y. Suppose, for example, that the potential energy V is expressed in terms of y. Then for equilibrium

$$\frac{dV}{dy} = 0$$

For stability the rate of change of dV/dy must be positive or, in other words,

$$\frac{d^2V}{dy^2} \text{ must be positive}$$

The ball at the top of the hill has maximum potential energy. A

small displacement decreases the potential energy or changes dV/dy from zero at the equilibrium position to some negative value at the displaced position. For unstable equilibrium

$$\frac{d^2V}{dy^2} \text{ is negative}$$

Finally it may be readily seen that, if the equilibrium is neutral, then

$$\frac{d^2V}{dy^2} \text{ is zero}$$

Thus the stability of a conservative system in equilibrium depends on whether the value of d^2V/dy^2 is positive, negative, or zero.

PROBLEMS

1. Along the sides of a triangle X, Y, Z taken in order, forces proportional to the lengths XY, YZ, $3ZX$ are acting. Show that the resultant force acts at a point A along YZ extended, where $ZA = YZ/2$, and also show that its magnitude is $2ZX$ and that its direction is parallel to ZX.

2. Along the sides of an equilateral triangle A, B, C taken in order, forces of 4, 6, and 8 lbw respectively are acting. Show that the resultant of these forces has a magnitude of $2\sqrt{3}$ lbw and acts at a point D along AC extended, where $CD = 2AC$. Show that the resultant force acts in a direction making an angle of 30° with AC.

3. Forces of 6 lbw and -6 lbw act perpendicular to the X axis through points having x coordinates of 0 and 2 respectively, and a third force of 12 lbw acts through the point having an x coordinate of 5 in a direction making an angle of 30° with the positive X axis. Show that the resultant of this couple and the single force is a force of 12 lbw acting through the point having an x coordinate of 3 in a direction making an angle of 30° with the positive X axis.

4. Give a proof showing that the moment of a force about any point is equal to the sum of the moments of its components about the same point.

5. A balance is made up of a rigid rod free to rotate about a point not at the center of the rod. It is balanced by unequal weight pans placed at each end of the rod. When an unknown mass m is placed in the left-hand pan it is balanced by a mass m_1 placed in the right-hand pan, and similarly when a mass m is placed in the right-hand pan it is balanced by a mass m_2 in the left-hand pan. Show that $m = \sqrt{m_1 m_2}$ and indicate any assumptions that you make in doing this.

6. An equilateral triangle is made up of three uniform rods, each of weight W, freely jointed at their ends. The triangle is supported on a fixed peg placed at the center of one of the rods so that this rod is horizontal. Find the magnitude and direction of the reactions at the hinges.

7. A gate weighing 50 lbw is hung on two hinges, the lower one supporting the whole of the weight. The hinges are 4 ft apart vertically, and the center of gravity of the gate is 5 ft from the line of the hinge. Find the forces exerted by the hinges on the gate.

8. A stepladder is made up of two uniform identical ladders hinged at the top. The ladders each have a length of 8 ft and a weight of 15 lbw. They are maintained at an angle of 60° with the horizontal by a rope connecting the mid-points of

the ladders. If a man weighing 150 lbw stands on one of the ladders 2 ft from the top, find the tension in the rope and the force at the hinge when:

a. The ladder is placed on a smooth floor.

b. The force of friction on the ladder holding the man is that associated with a coefficient of friction of 0.2 at the floor. Find the force of friction acting at the foot of the other ladder.

9. A uniform ladder having a length of 12 ft and a weight of 30 lbw rests against a vertical wall at an angle of 60° with the horizontal If the coefficient of friction at the floor is 0.45 and a man weighing 200 lbw can walk 10 ft up the ladder before slipping takes place, find the force and the coefficient of friction at the wall.

10. The boom of a derrick is hinged at one end to a vertical post and at the other end carries a load of 3000 lbw. A cable attached to the vertical post slopes downward, at an angle of 60° with the vertical, and is attached to the end of the boom where the load is placed If the boom has an upward slope of 45°, find the tension in the cable and the force at the hinge.

11. A uniform ladder having a length of 20 ft and a weight of 50 lbw rests at an angle of 60° with the horizontal against a smooth vertical wall If the coefficient of friction between the ladder and the ground is 0.4, find how far up the ladder a 150-lbw man can go before the ladder starts to slip

12. From Fig. 8.10 show that the force triangle for the section of cord OP is PMN. Using this figure derive Eqn. 8.5.

13. If the coordinates of the point of support B of the cable in Fig. 8.10 are h and v, show that the horizontal tension in the cable is $Wh/4v$, where W is the total weight of the suspended load. Also show that the tension at the supports is

$$\frac{W}{4v}\sqrt{h^2 + 4v^2}$$

14. In Fig. 8.11 for the catenary let T be the tension in the cable at any point x, y, having a length s from C. Let $CO = a$ and the weight per unit length of the cable be w, and the vertical height of the point x, y above the vertex C, or the sag, be y'. Then, if a new variable $z = x/a$ is introduced, show that

$$T = wy = \frac{wx \cosh z}{z},$$

$$s = \frac{x \sinh z}{z}$$

and

$$y' = \frac{x(\cosh z - 1)}{z}$$

(These relationships are useful in solving the following problems.)

15. A uniform cable 120 ft long and weighing 0.15 lbw per ft is supported between two posts 90 ft apart and from the same vertical height. Find:

a. The horizontal tension in the cable.

b. The tension at the supports.

c. The sag of the mid-point of the cable below the horizontal level of the supports.

16. A telephone line has posts 200 ft apart. It is desired to have a maximum allowable tension of 80 lbw, using wire of 0.1 lbw per ft. If the line is supported from two points at the same horizontal level, find:

a. The tension at the middle point of the line.

b. The sag of the line at the middle below the horizontal level of the supports.

c. The total length of the telephone line between the posts.

17. If the wire of the telephone line in problem 16 expands by 0.2 per cent on a hot day, find the new tension at the supports.

18. Find the maximum tension in a wire having a weight of 0.5 lbw per ft which, when suspended between two horizontal points 100 ft apart, has a sag of 1 ft at the middle.

19. A single span of cable crosses a valley 1000 ft wide. The cable weighs 1.5 lbw per ft and is supported from points at the same horizontal level. If the maximum tension is 1800 lbw, find:

a. The sag.

b. The total length of cable between the supports.

20. An electric cable is supported from two posts 90 ft apart at points having a vertical separation of 18 ft. If the cable weighs 0.4 lbw per ft and sags below the lowest point of support by 1 ft, find the tension at each of the supports.

21. A boy, holding the end of the string of a kite which is flying in the air, exerts a force of 2 lbw at an angle of 30° above the horizontal. If the string weighs 0.05 lb per 100 ft and the boy has 800 ft of string extended, find:

a. The force exerted by the kite.

b. The vertical height of the kite.

(Neglect the weight of the kite.)

22. A suspension bridge has a total weight of 10 tons. The horizontal distance between the supports of the cable (assumed to be of negligible weight) is 200 ft, and the vertical distance between the lowest and highest points of the cable is 20 ft. Find the force exerted by the cable on the supports.

23. A horizontal rough peg whose diameter is 6 in. has a light cord hung over it on one side of which hangs a weight of 10 lbw. If the coefficient of friction between the cord and the peg is 0.6, find the downward vertical force that must be exerted on the free end of the cord in order to raise the weight at a constant speed.

24. Masses of 500 gm and 100 gm are placed one at each end of a light, inextensible cord passing over a rough fixed peg having a coefficient of friction of 0.4. Find:

a. The tension in the cord on each side of the peg.

b. The acceleration of the system.

25. A uniform beam weighing 50 lb is at rest with its upper end against a smooth vertical wall and with its lower end on a smooth horizontal table. The lower end is prevented from slipping by a horizontal cord in which there is a tension of 100 lb. Find the angle the beam makes with the vertical, from:

a. The general conditions of equilibrium.

b. The principal of virtual work.

c. The equilibrium and potential energy theorem.

26. Six uniform rods of the same length and of weight W are freely jointed at their ends so as to form a hexagon. This hexagon is then placed in a vertical plane with the lower horizontal rod resting on a horizontal table. The hexagon is kept in position by a vertical cord attached to the mid-point of the upper rod. Prove that the tension in the cord is $3W$.

27. Four uniform rods of the same length and of weight W are freely jointed at the ends to form an equal-sided parallelogram or rhombus. This is held in a vertical plane with one corner resting on a horizontal table and the diagonal from this corner vertical. A horizontal cord placed along the other diagonal holds the

rhombus in position. If θ is the angle the vertical diagonal makes with its sides, show that the tension in the cord T is given by

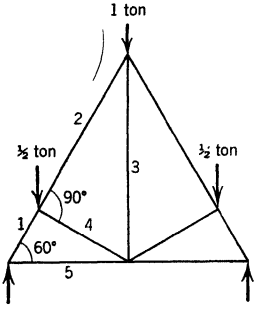
$$T = 2W \tan \theta$$

28. Four uniform rods of the same length and having a weight W are freely jointed at their ends to form a rhombus. This is suspended in a vertical plane with one corner of the rhombus resting on a fixed peg. A light rod joins the mid-points of the upper two rods of the rhombus so as to hold it in position with the upper angle of the rhombus equal to 2θ. Prove by the principle of virtual work that the force in the rod is $4W \tan \theta$.

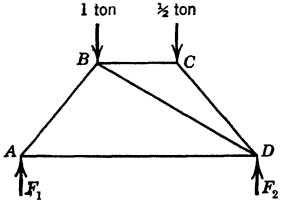
29. Three uniform rods of length L and weight w are freely joined at the upper end to form a tripod. This is held in position by joining the middle of each rod to the middle of the next by means of horizontal strings of length l. The tripod is placed on a smooth floor with a camera of weight W resting on top of the tripod. Prove by the principle of virtual work that the tension T in the strings is

$$T = \frac{(4W + 6w)l}{3\sqrt{9L^2 - 12l^2}}$$

30. A symmetrical system of light rods freely joined together and supported on two pillars is shown in the figure. Two loads of 0 5 ton each and one load of 1 ton are placed as shown. Find the forces in the various members marked 1, 2, 3, 4, 5.



PROBLEM 30



PROBLEM 31

31. A framework $ABCD$ is made up of light rods with a brace at BD as shown in the accompanying figure. Given $AB = CD = 3$ ft, $BC = 2$ ft, and $AD = 6$ ft, and the fact that BC is symmetrically placed with respect to AD. Loads of 1 and 0.5 ton are placed at B and C respectively. Find the forces of support F_1 and F_2 and also the forces in the rods AB, BC, CD, DA, BD.

32. A simple triangular framework ABC is made up of three identical, uniform rods of weight W freely joined at the ends and placed in a vertical plane. The horizontal rod AC is supported at its ends A and C on pillars. Find the forces at the pillars and the forces at the joints A, B, C.

33. A triangular framework ABC is made up of three identical, uniform rods of weight W freely joined together at the ends. It is supported in a vertical plane on a horizontal fixed peg placed at the middle of the horizontal side AB with the joint C vertically below the peg. Find the reactions at the joints A and C.

34. An electrostatic voltmeter is a device for measuring the direct-current voltage between two sets of plates, one of which can move with respect to the other. The two sets of plates form a variable condenser whose capacitance C in farads can be found from

$$C = a + b\theta$$

The movable set of plates has a pointer attached which can turn on an axis about which a flat, coiled spring is wound. The torque developed by the electrostatic forces is balanced by the mechanical torque $k\theta$ developed in the spring. If $a = 5 \times 10^{-12}$ farad, $b = 6 \times 10^{-11}$ farad per radian, and $k = 10^3$ dyne cm per radian, find the deflection produced by a voltage of 2000 volts between the plates. Given the electrostatic energy E of a capacitance of C farads and a potential difference or voltage of V volts,

$$E = \tfrac{1}{2}CV^2 \text{ joules}$$

Then show from the principle of virtual work that the torque $k\theta$ is given by

$$k\theta = \frac{dE}{d\theta}$$

9 · HYDROSTATICS AND HYDRODYNAMICS

9.1 The Three States of Matter

The three states of matter—solid, liquid, and gaseous—are characterized by very different physical properties which for our present purposes can be summarized in the following manner. Large forces are required to change the shape or volume of a solid. With liquids, although a large force is required to change their volume, practically no force is required to change their shape. In other words, liquids have a large bulk modulus but practically no rigidity or shear modulus. Almost no force is required to change the shape of a gas, and only a small force to change the volume. Thus gases have practically no rigidity or shear modulus and only a small bulk modulus.

When we attempt to place every substance into one of these three classes, we experience some difficulty. For example, ice is considered to be a solid. However, this substance is known to flow very slowly downhill under the action of gravity. The flow of glaciers is well known to geologists. Since the flow of so-called solids is rarely met, we need not be concerned with them here.

In many cases the properties of gases and liquids are similar and can be discussed under the single heading of fluids. For the present we shall deal with perfect fluids, which exhibit no forces opposing changes in shape, or have no shear modulus. In a later section this assumption will be modified and the results discussed under the subject of viscosity. It will be further assumed that solids, liquids, and gases are continuous throughout their volume. This, of course, contradicts the atomic or molecular theory of matter. However, it is easy to show that any very small element of volume contains a very large number of molecules; for example, 1 mm^3 of air contains about 3×10^{16} molecules. We may also neglect the forces between the individual molecules and consider the matter to be continuous. We have already assumed a continuous theory of matter in the discussion of elasticity. The alternative would have been to explain the properties of substances in terms of intermolecular forces, which is extremely difficult. First we shall discuss the theory of fluids in equilibrium, that is,

hydrostatics, and later some of the phenomena exhibited by fluids in motion, that is, hydrodynamics.

9.2 Pressure in Hydrostatics

Let us consider a small plane area ΔA inside a perfect fluid at rest. A force ΔF is exerted by the fluid on one side of this area, and an equal and opposite force is exerted by the fluid on the other side. The pressure p at this area is defined as the force per unit area or

$$p = \frac{\Delta F}{\Delta A}$$

At the point at which this area is situated the pressure is given by

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA} \tag{9.1}$$

As will now be shown, this force is always at right angles to the area. Thus, when we consider the vessel shown in Fig. 9.1 as completely filled with a liquid of density ρ, the force exerted by the liquid on the plane sloping area must be at right angles to the area. By Newton's third law the force exerted by the sloping area on the liquid in contact with it must be at right angles to the area. Were this not so, the force could be resolved into components along and perpendicular to the sloping area. Any force parallel to this area would cause the liquid to move, and this would violate the condition that the liquid is at rest. Thus the force exerted on any plane area within a liquid at rest is normal to the area.

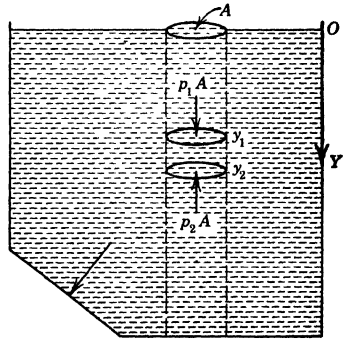


FIG. 9.1 Pressure in a liquid.

Now any plane area can be represented by a vector normal to it and having a magnitude proportional to the area. Thus the force exerted by a liquid on any area has the same direction as the outward drawn normal to the area. By writing Eqn. 9.1 as $d\mathbf{F} = \mathbf{n}_1 p\, dA$, where $\mathbf{n}_1$ is a unit vector along the outward drawn normal, it appears that the pressure p is a scalar quantity. Actually pressure is a tensor, though in the simple problems we can consider it to be a scalar just as we consider temperature to be.

Now imagine a right circular cylinder, drawn within the liquid, of area A as shown dotted in Fig. 9.1. Suppose that the Y axis has its origin at the surface of the liquid and is positive in a downward direction. Let us consider that two horizontal cross sections cut this cylinder at y_1 and y_2 to form a pillbox-shaped mass of liquid of height $(y_2 - y_1)$. If p_1 and p_2 are the pressures at the upper and lower surfaces respectively, then there is a downward force of p_1A exerted on the upper surface at y_1 and a corresponding upward force p_2A on the lower surface. The liquid in this pill box is in equilibrium under the action of the two surface forces on its ends and the body force of the weight of the liquid contained within it. All other forces exerted on the cylinder are horizontal and have no vertical components. Hence for equilibrium

$$p_2A - p_1A = A(y_2 - y_1)\rho g$$

or

$$(p_2 - p_1) = (y_2 - y_1)\rho g$$

If we let

$$y_2 - y_1 = dy \qquad \text{and} \qquad p_2 - p_1 = dp$$

then

$$\frac{dp}{dy} = \rho g \tag{9.2}$$

If the density ρ is a constant, then

$$p = \rho g y + C$$

where C is a constant of integration. Now suppose that at the surface of the liquid in the vessel the pressure is p_0, or at $y = 0$, $p = p_0$. Then

$$p = p_0 + \rho g y \tag{9.3}$$

The pressure increases linearly with downward depth in a liquid of constant density.

The common units of pressure are dynes per square centimeter, pounds weight per square foot, and pounds weight per square inch. In meteorological work a unit of pressure called the *bar* is introduced. It is equivalent to a million dynes per square centimeter. Since normal atmospheric pressure is the pressure equal to that exerted by a column of mercury 76 cm high and 13.6 gm per cm^3 in density, it follows from Eqn. 9.3 that this atmospheric pressure is

$$p_{atm} = 13.6 \times 980 \times 76$$
$$= 1.013 \times 10^6 \text{ dynes per cm}^2$$
$$= 1.013 \text{ bars}$$

For a large mass of gas such as exists in our atmosphere the density ρ is not a constant independent of the height above sea level and the pressure is not a simple linear function of the height.

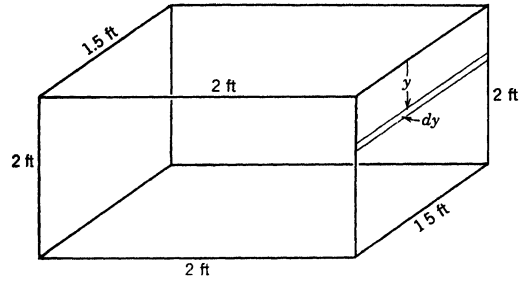


FIG. 9.2 Forces exerted by a liquid on the sides of a vessel.

In order to consider the forces exerted by a liquid let us think of a vessel filled with water as shown in Fig. 9.2. The problem is to find the forces exerted by the water on the various portions of the vessel. The force on the base is

$$F = pA$$
$$= hg\rho A$$
$$= 2 \times 62.4 \times 2 \times 1.5$$
$$= 374.4 \text{ lbw}$$

We have taken ρg as equal to 62.4 lbw per ft^3 for water.

To determine the force on any of the sides consider a horizontal strip along one of the sides at a depth y from the top and a width dy. The pressure at this strip produced by the water is $y\rho g$, and the force exerted on such a strip is

$$F = 1.5 \int_0^2 yg\rho \, dy = \left[1.5g\rho \frac{y^2}{2} \right]_0^2$$
$$= 1.5 \times 62.4 \times \frac{2^2}{2} = 187.2 \text{ lbw}$$

Next let us determine the vertical height at which this force acts. Suppose that this is a distance y_0 from the top of the vessel. Then the moment of the resultant force about the top edge of this wall as the axis must be equal to the sum of the moments of all the strips making up this wall about the same axis. Thus

$$y_0 = \frac{1.5\rho g \int_0^2 y^2\,dy}{1.5\rho g \int_0^2 y\,dy} = \frac{8}{3} \times \frac{2}{4} = \frac{4}{3} \text{ ft}$$

Therefore the point at which this resultant force acts, sometimes called the center of pressure, is $\frac{4}{3}$ ft from the top and $\frac{2}{3}$ ft from the bottom of the liquid.

9.3 Archimedes' Principle

An important principle in hydrostatics was first presented by Archimedes (287–212 B.C.). This principle states that when a body is immersed in a fluid there is a resultant upward vertical force exerted on the body by the fluid equal to the weight of the fluid displaced

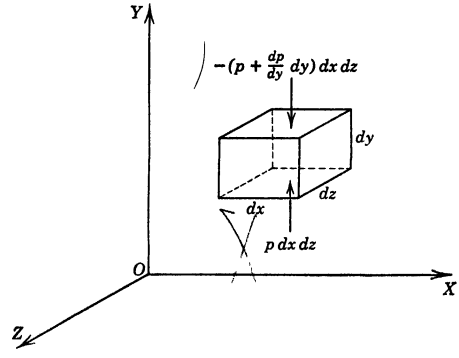


FIG. 9.3 Vertical forces exerted on a parallelopiped of a body placed in a liquid.

by the body. In order to prove this principle consider an element of volume dx, dy, dz of a body immersed in a fluid of density ρ. On the four vertical sides of this element of volume, Fig. 9.3, the forces are all horizontal and have no upward component. But there are vertical forces on the horizontal faces of the element of volume. The difference between the upward force on the lower face and the downward force on the upper face is

$$p\,dx\,dz - \left(p + \frac{dp}{dy}\,dy\right) dx\,dz = -\frac{dp}{dy}\,dy\,dx\,dz$$

Now dp/dy is the rate of increase in pressure with height and from Eqn. 9.2 is

$$\frac{dp}{dy} = -g\rho$$

since we have here chosen the Y axis to be positive upward. Thus the resultant upward force on the element of volume is

$$dF_y = g\rho\,dx\,dy\,dz$$
$$= g\rho\,dV$$

The buoyant force on a body of volume V immersed in a fluid of density ρ is then

$$F_y = \int_0^V g\rho\,dV = g\rho V \tag{9.4}$$

Now ρV is the product of the density of the fluid and the volume of the body and is therefore the mass of fluid displaced by the body. Thus Eqn. 9.4 is a mathematical statement of *Archimedes' principle.*

When a body is floating in a liquid, the weight of the liquid displaced must equal the weight of the body. In other words, for a body to float in a liquid the average density of the body must be less than the density of the liquid. This of course must be the case for a ship whose hull is made of steel. Not only must the ship float, but it must also be stable in the water. If the ship is disturbed from its equilibrium position, there must be present forces that when brought into play will tend to return the ship to its equilibrium position.

The upward buoyant force on the ship acts through a point in the ship called the *center of buoyancy* which is at the center of gravity of the displaced liquid before its displacement. In order to show that this is true let us consider what would happen if the ship were removed and the displaced water were back in its former position. The water surrounding the ship is in equilibrium with either the ship or with the water replacing the ship. In the latter case the force is exerted at the center of gravity of the displaced water and consequently the center of gravity of the displaced water must be the same as the point in the ship where the buoyant force acts.

Consider the ship shown in Fig. 9.4*a*, whose center of gravity is at G and whose center of buoyancy is at B. In Fig. 9.4*b* the ship is shown slightly displaced so that the center of buoyancy is now at B'. Let

the vertical line through B' intersect the line of symmetry BG at M, called the *metacenter*. Figure 9.4*b* shows that if M is above G the equilibrium is stable; if M is below G the equilibrium is unstable; and if M coincides with G the equilibrium is neutral. The distance GM is called the *metacentric height* h. Thus the equilibrium is stable, unstable, or neutral, according to whether h is positive, negative, or zero.

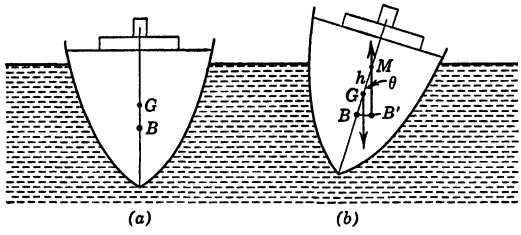


FIG. 9.4 Equilibrium of a floating body.

In practice the point M and the metacentric height h depend on the angle of displacement θ (see Fig. 9.4*b*), but for relatively small displacements the point M can be considered to be fixed. We shall now determine the period of roll of the ship in terms of the metacentric height h, assuming it to be positive. Consider the motion about the center of mass of the ship. The torque about G is

$$V\rho gh \sin \theta$$

since the buoyant force is $V\rho g$ where V is the volume of displaced water whose density is ρ. This torque produces an angular acceleration $\ddot{\theta}$ about the center of mass of the ship given by

$$I\ddot{\theta} = -V\rho gh \sin \theta$$

where I is the moment of inertia of the ship about a horizontal axis through its center of mass.

If θ is sufficiently small so that it may be substituted for $\sin \theta$, then

$$\ddot{\theta} = -\frac{V\rho gh\theta}{I}$$

This is the equation for simple harmonic motion whose period P is

$$P = 2\pi\sqrt{\frac{I}{V\rho gh}}$$

Thus the period is inversely proportional to the square root of the metacentric height h. A very stable ship has a large metacentric height and consequently a relatively small period. This may be very safe but gives somewhat uncomfortable riding.

The positions of G and M depend to a large extent on the load the ship is carrying. When a ship has no pay load, it usually carries ballast purely for reasons of safety. The angle through which a ship can roll safely depends, of course, on how the ship is built but for most ships is of the order of 40°. Long before this angle of roll is reached, most passengers would be sure the safety angle had been passed.

9.4 Hydrodynamics. The Equation of Continuity

The problems associated with fluids in motion can be very complex, so we shall analyze only some of the simplest of them. Let us consider first what is meant by the stationary or *steady flow* of a fluid. In this type of flow the velocity of the fluid at any one point is always the same. Of course the velocity will probably be different at different points along a line of flow. Any small element of volume of the fluid follows a particular path called a *stream line*. The tangent at any point in the stream line gives the direction of the velocity of the fluid at the point. If V is the velocity and ds is a distance measured along the stream line whose components along the X, Y, Z axes are dx, dy, dz respectively, then the equation of the stream line in differential form is

$$\frac{ds}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{9.5}$$

where u, v, w are the components of V along the X, Y, Z axes respectively.

The stream lines in a liquid can be made visible by inserting in the liquid a very small amount of colored liquid and following the lines taken by the colored liquid. It follows from the definition of stream lines that an element of fluid can never pass from one stream line to another. Suppose that the flow of the fluid is such that the velocities of the fluid elements over any cross section perpendicular to the flow are constant. The stream lines then form a tube of flow which may be considered a bundle of stream lines. Let Fig. 9.5 represent a tube of flow which might be an ideal liquid flowing in a tube with a constriction. Over the cross-sectional area A_1 the velocity is v_1, and similarly over A_2 it is v_2. If ρ_1 is the density of the fluid at A_1, then $A_1 v_1 \rho_1$ represents the mass of fluid crossing the area A_1 each second. Similarly, if ρ_2 is the density of the fluid at the cross section A_2, then

$A_2v_2\rho_2$ is the mass of fluid crossing the area A_2 per second. If there are no sources or sinks within the volume between A_1 and A_2, then the mass of fluid flowing into the tube per second at A_1 is equal to

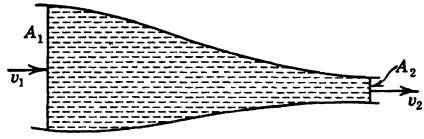


FIG. 9.5 The principle of continuity in a liquid.

the mass of fluid flowing out of the tube per second at A_2. This is a statement of the principle of conservation of matter. Thus we have

$$A_1v_1\rho_1 = A_2v_2\rho_2 \tag{9.6}$$

If the fluid is incompressible or if its density is constant, as is very approximately true for all liquids, then

$$A_1v_1 = A_2v_2 \tag{9.7}$$

Thus where the cross section is large the velocity is small, or "still water runs deep."

The principle of continuity expressed in Eqns. 9.6 and 9.7 can be given in a differential form. Let V be the velocity at some point x, y, z with components u, v, w parallel to the X, Y, Z axes respectively.

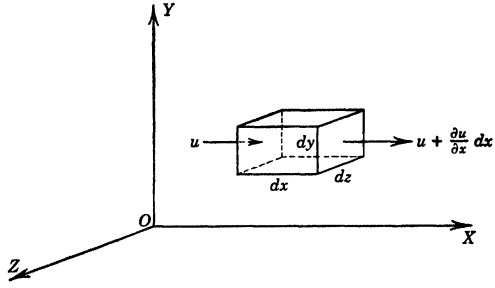


FIG. 9.6 The principle of continuity for a small parallelopiped of liquid.

Around the point x, y, z construct a parallelopiped whose sides have the lengths dx, dy, dz respectively, Fig. 9.6. Consider the fluid flowing through this element of volume. Into the left-hand face whose area is $dy\,dz$, perpendicular to the X axis, there is a mass of fluid flowing

per second equal to $\rho u\, dy\, dz$. Through the right-hand face there is a mass of fluid flowing out per second equal to

$$\left[\rho u + \frac{\partial(\rho u)}{\partial x}\, dx\right] dy\, dz$$

The net outward flow per second through the parallelopiped in the X direction is

$$\frac{\partial(\rho u)}{\partial x}\, dx\, dy\, dz$$

Similar expressions hold for the net outward flow in the Y and Z directions. Hence the total excess mass of fluid flowing per second out of the parallelopiped over that entering each second is

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dx\, dy\, dz$$

This expression gives the rate of loss of mass of fluid within the volume $dx\, dy\, dz$. Hence, if there are no sources or sinks within this volume, the equation of continuity can be written in the following manner:

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dx\, dy\, dz = -\frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

or dividing by $dx\, dy\, dz$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{9.8}$$

If the density of the fluid is a constant, then the *equation of continuity* becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9.9}$$

Equations 9.8 and 9.9 can be written in an abbreviated form by means of the vector differential operator del written as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the X, Y, Z axes. The velocity $\mathbf{V}$ of the fluid has the components u, v, w so that

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

Writing the dot product of ∇ and $\rho\mathbf{V}$ we may express the equation of continuity, Eqn. 9.8, as follows:

$$\nabla \cdot (\rho\mathbf{V}) + \frac{\partial \rho}{\partial t} = 0$$

If the density ρ is constant as in Eqn. 9.9, then this may be written

$$\nabla \cdot \mathbf{V} = 0$$

The quantity $\nabla \cdot \mathbf{V}$, del dot V, is called the divergence of the vector V. It is a measure of how much more of a vector quantity leaves a unit volume than enters the unit volume per second. The equation of continuity for electric charges is similar to that given above. If I is the electric current or the amount of electric charge flowing past any point per second and ρ is the volume density of charge at the point, then the law of conservation of electric charge requires that

$$\nabla \cdot \mathbf{I} + \frac{\partial \rho}{\partial t} = 0$$

9.5 Bernoulli's Theorem

About 1738 Daniel Bernoulli applied the law of conservation of mechanical energy to the motion of an incompressible liquid having a steady flow. In order to prove Bernoulli's theorem consider a tube

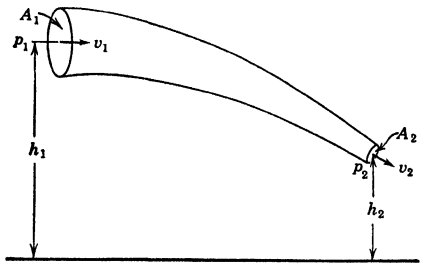


FIG. 9.7 Bernoulli's principle for a liquid.

of flow having a pressure p_1 and a velocity v_1 over the cross-sectional area A_1, and a pressure p_2 and velocity v_2 over the area A_2, Fig. 9.7. Let the mean height of A_1 above some horizontal level be h_1, and similarly let h_2 be the mean height of A_2 above this level. If the area A_1 is larger than A_2, then for an incompressible liquid Eqn. 9.7 shows that the velocity v_1 is less than v_2. Between A_1 and A_2 the

flow of the liquid is speeded up or accelerated. This acceleration comes about by a reduction in the pressure between A_1 and A_2 in the liquid and a reduction in the vertical height of the liquid. The liquid gains kinetic energy at the expense of the pressure energy and the potential energy. Bernoulli's theorem gives a quantitative relationship between these forms of energy.

In a small time interval Δt a mass Δm of liquid of density ρ passes through the area A_1 into the tube where

$$\Delta m = \rho A_1 v_1 \, \Delta t$$

In this same time interval this same mass Δm passes out through A_2. This assumes that there are no sources or sinks between A_1 and A_2. Thus

$$\Delta m = \rho A_1 v_1 \, \Delta t = \rho A_2 v_2 \, \Delta t \tag{9.10}$$

The work done by gravity on this mass Δm as it moves from A_1 to A_2 is

$$g \, \Delta m(h_1 - h_2)$$

The work done by the pressure forces $p_1 A_1$ and $p_2 A_2$ respectively on the mass Δm having the volume $A_1 v_1 \, \Delta t = A_2 v_2 \, \Delta t$ is

$$\int_1^2 p \, dV = p_1 A_1 v_1 \, \Delta t - p_2 A_2 v_2 \, \Delta t$$

The change in the kinetic energy of the mass Δm in moving from A_1 to A_2 is

$$\tfrac{1}{2} \Delta m(v_2^2 - v_1^2)$$

Assuming a perfect fluid and no dissipative forces, it follows from the principle of conservation of mechanical energy that the work done on the mass Δm is equal to the increase in the kinetic energy of this mass. Thus from the principle of conservation of energy we have

$$g \, \Delta m(h_1 - h_2) + p_1 A_1 v_1 \, \Delta t - p_2 A_2 v_2 \, \Delta t = \tfrac{1}{2} \Delta m(v_2^2 - v_1^2)$$

Dividing this equation by Δm, which is given by Eqn. 9.10, it follows that

$$g(h_1 - h_2) + \frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2)$$

or

$$\frac{p_1}{\rho} + gh_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + gh_2 + \frac{1}{2} v_2^2$$

Thus along a stream line the quantity

$$\frac{p}{\rho} + hg + \frac{v^2}{2} = \text{Constant} \tag{9.11}$$

The terms in this equation can be physically interpreted in the following manner: p/ρ is the pressure head or the potential energy per unit mass of the liquid due to the pressure; hg is the elevation head or the potential energy per unit mass of the liquid due to gravity; and $v^2/2$ is the kinetic energy per unit mass of the liquid.

9.6 The Eulerian Equations of Motion of a Liquid

In this mode of analysis, first given by Euler about 1755, we choose a particular point in space and try to determine the velocity, density, and pressure of the liquid at that point for all instants of time. First we find the expression for the acceleration at a point in the liquid.

Suppose that a fluid element is at P, the point x, y, z, at time t, and in an interval of time dt moves to Q, the point whose coordinates are $x + dx$, $y + dy$, $z + dz$. Let V be the velocity of the fluid element at P which has components parallel to the coordinate axes of u, v, w respectively. Then each of the components u, v, w are functions of the independent variables x, y, z, and t. The increase in the X component of the velocity of the fluid element in moving from P to Q is

$$du = \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz \tag{9.12}$$

The X component of the velocity u is equal to dx/dt. Similarly

$$v = \frac{dy}{dt} \qquad w = \frac{dz}{dt}$$

Thus from Eqn. 9.12 the acceleration of the fluid element at P in the X direction is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{9.13}$$

Similar expressions hold for the components of acceleration along the Y and Z axes. In moving from P to Q there is a step forward in time and a step forward in distance. The expression for the acceleration, Eqn. 9.13, contains these two factors: the first term on the right is due to the time factor; the remaining three terms are due to the space factor. Notice that the partial derivative of the component velocity u with respect to time is the rate of change of the velocity u with respect to time but not with respect to distance.

In order to relate the acceleration to the forces we shall choose a small rectangular parallelopiped drawn about the point P in the liquid, shown in Fig. 9.8. Let p be the pressure, ρ the density, and $\mathbf{F}$ the extraneous *force per unit mass* at the point P. Suppose that the force $\mathbf{F}$ has components F_x, F_y, F_z along the X, Y, Z axes respectively.

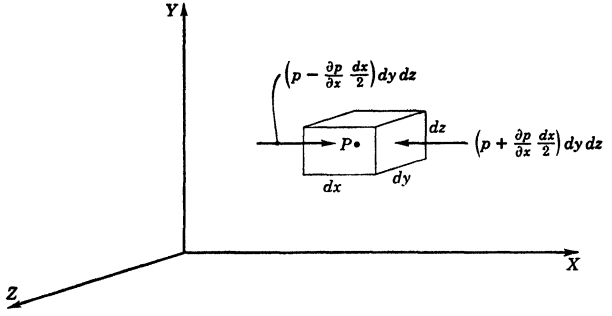


FIG. 9.8 Horizontal forces on a small parallelopiped of liquid.

The rate of change of momentum of the fluid element in the X direction is

$$\rho\, dx\, dy\, dz \frac{du}{dt}$$

The body or extraneous force in the X direction on this fluid element is

$$\rho\, dx\, dy\, dz\, F_x$$

Finally there is a force due to the pressure. On the left-hand face of the fluid element, which has an area $dy\, dz$, there is a force of

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz$$

Similarly on the right-hand face there is a force of

$$\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz$$

The resultant force in the X direction due to the pressure change is

$$-\frac{\partial p}{\partial x}\, dx\, dy\, dz$$

Thus from Newton's second law the equation of motion of the point P

in the fluid in the X direction is

$$\rho\, dx\, dy\, dz \frac{du}{dt} = \rho\, dx\, dy\, dz\, F_x - dx\, dy\, dz \frac{\partial p}{\partial x}$$

and, dividing by $\rho\, dx\, dy\, dz$, it follows that

$$\frac{du}{dt} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{9.14}$$

using Eqn. 9.13. Similar expressions can be immediately written for the Y and Z components. If the force $\mathbf{F}$ per unit mass can be derived from a potential Ω which is independent of time, then

$$F_x = -\frac{\partial \Omega}{\partial x} \qquad F_y = -\frac{\partial \Omega}{\partial y} \qquad F_z = -\frac{\partial \Omega}{\partial z}$$

and the equation of motion, Eqn. 9.14, may be written

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{9.15}$$

Similar equations can be written for dv/dt and dw/dt.

Now these equations are a starting point for many relatively difficult derivations which we shall have to omit here. Instead we shall apply the equations of motion to the derivation of Bernoulli's theorem.

Let us transform Eqn. 9.15 to an expression for steady motion along a stream line in a direction s. At the point P the velocity V of the fluid element is tangential to the stream line and V is a function of s and t. The acceleration along the stream line may be obtained from Eqn. 9.13 and is given as

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}$$

Now in the Bernoulli theorem the motion is considered to be steady, or the velocity at any point in space is constant. Hence for steady flow

$$\frac{\partial V}{\partial t} = 0$$

and the acceleration along the stream line is

$$\frac{dV}{dt} = V\frac{\partial V}{\partial s}$$

Similarly the resultant force per unit mass at the point P along the

stream line can be obtained from Eqn. 9.15 and is

$$-\frac{\partial \Omega}{\partial s} - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

Thus the equation of motion for steady flow along a stream line is

$$V\frac{\partial V}{\partial s} = -\frac{\partial \Omega}{\partial s} - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{9.16}$$

By integration of Eqn. 9.16 we have

$$\int \frac{dp}{\rho} + \Omega + \frac{V^2}{2} = \text{Constant}$$

If the fluid is incompressible so that ρ is a constant, then for steady flow along a stream line we have Bernoulli's theorem:

$$\frac{p}{\rho} + \Omega + \frac{V^2}{2} = \text{Constant} \tag{9.17}$$

In the earlier derivation of Bernoulli's equation, section 9.5, we assumed the extraneous force per unit mass to be that due to gravity. Hence the potential energy per unit mass Ω is given by

$$\Omega = gh$$

where h is measured vertically in the Y direction. Thus Eqn. 9.17 becomes identical with Eqn. 9.11.

9.7 Applications of Bernoulli's Theorem

a. **The Venturi Meter.** This application of Bernoulli's theorem is extensively used for measuring the velocity of liquids and gases. The

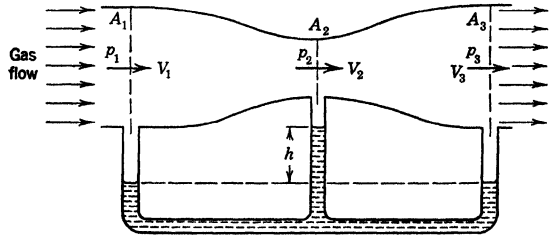


FIG. 9.9 The venturi meter for the velocity of a gas.

venturi meter usually consists of a horizontal pipe with a constriction in it. The pressure in the wide and the constricted portions can be measured by means of liquid manometers, as shown in Fig. 9.9, if gas

is flowing in the tube. If a liquid is flowing in the tube, the manometers will consist of three vertical tubes attached to the top of the main tube. If there is no friction between the tube and the moving fluid and the cross-sectional areas A_1 and A_3 are equal, then the pressure p_1 is equal to p_3 and the velocity V_1 is equal to V_3. It thus follows that the liquid in the manometer tubes at A_1 and A_3 stands at the same level. From the equation of continuity, Eqn. 9.7, the velocities and cross sections at the wide and constricted portions are given by

$$A_1V_1 = A_2V_2 \tag{9.7}$$

Bernoulli's theorem, Eqn. 9.11, assuming stream-line flow, applies to the venturi tube. From this and the relation $h_1 = h_2$ we have

$$\frac{1}{2}V_1^2 + \frac{p_1}{\rho} = \frac{1}{2}V_2^2 + \frac{p_2}{\rho}$$

where ρ is the density of the fluid and is assumed to be constant. Substituting for V_2 from Eqn. 9.7 in the above equation it follows that the velocity V_1 is given by

$$V_1 = A_2\sqrt{\frac{2(p_2 - p_1)}{\rho(A_2^2 - A_1^2)}}$$

If the liquid in the manometer tubes in Fig. 9.9 has a density ρ' and the difference in levels of the liquid in the two tubes is h, then

$$p_2 - p_1 = h\rho'g$$

In using the venturi meter to measure the speed of a gas there is some error introduced due to the difference in density of the gas in the wide and constricted portions of the tube. This is less than 1% if the velocity of the gas does not exceed about 50 meters per second.

In the above example we have assumed that there is no friction between the moving liquid and the stationary tube. This of course is not possible in practice. Its effect can be shown by allowing a liquid to flow through a horizontal tube of constant diameter as shown in Fig. 9.10. The pressure in the liquid decreases in the direction of the motion, as indicated by the height of the liquid in the manometer tubes. From Eqn. 9.17 the energy per unit mass of the liquid at A is

$$\frac{p_A}{\rho} + gh_A + \frac{v_A^2}{2}$$

Similarly for the point B the energy per unit mass is

$$\frac{p_B}{\rho} + gh_B + \frac{{v_B}^2}{2}$$

If there were no energy loss along the tube, these two expressions would be equal according to Bernoulli's theorem. However, there

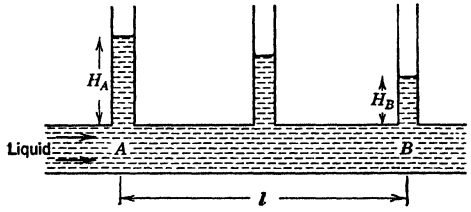


FIG. 9.10 Energy loss in a pipe carrying a liquid.

is an energy loss. Let E be the energy loss per unit mass per unit length along the tube. The energy loss per unit mass of fluid between A and B is El and is given by

$$\frac{p_A}{\rho} + gh_A + \frac{{v_A}^2}{2} - \frac{p_B}{\rho} - gh_B - \frac{{v_B}^2}{2} = El$$

From the conditions of the problem it follows that

$$h_A = h_B \qquad \text{and} \qquad v_A = v_B$$

Hence

$$\frac{p_A - p_B}{\rho} = El$$

If H_A is the height of the liquid in the manometer tube A and H_B the height in manometer tube B, then

$$p_A = H_A g\rho \qquad p_B = H_B g\rho$$

The energy loss per unit mass per unit distance along the tube is then

$$E = (H_A - H_B)\frac{g}{l}$$

b. The Pitot Tube. Another device for measuring the speed of fluids is the pitot tube. One type for measuring the speed of liquids is shown in Fig. 9.11. In this the opening A is perpendicular to the flow and the opening B faces the flow. As a result the levels of the

liquid in the two arms are separated by a height h. Along the stream line, which is in line with the tube B, the velocity of the liquid falls rapidly to zero and the manometer reading gives the value of $p + \rho V^2/2$. The tube A, past which the stream lines flow, measures the static

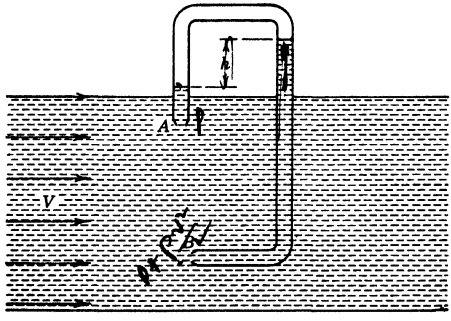


FIG. 9.11 The pitot tube for a liquid.

pressure p in the liquid. Since the two manometer tubes are connected together and have a difference in level of h, then it follows that

$$g\rho h = \frac{\rho V^2}{2}$$

or

$$V = \sqrt{2gh}$$

A pitot tube is frequently attached to an airplane for measuring the velocity of the air relative to the plane, or the so-called airspeed of the plane. It is represented diagrammatically in Fig. 9.12. It is placed

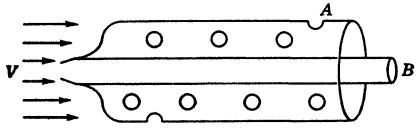


FIG. 9.12 The pitot tube for a gas.

on the leading edge of a wing or in the nose. The air rushes through tube B, giving the value of $p + \rho V^2/2$; a second tube A has openings on its sides giving the value of the pressure p. The two tubes are usually connected to a differential aneroid barometer which measures the difference in these pressures or $\rho V^2/2$. By proper calibration the

pointer on the aneroid gives the velocity V of the plane, though of course this is only possible for one particular value of the density ρ of the air. Since the density varies with altitude, this is hardly an accurate instrument for measuring airspeed.

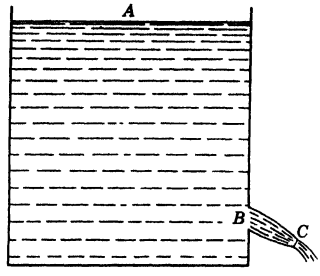


FIG. 9.13 Torricelli's theorem for a liquid emerging from a hole in a vessel.

c. Torricelli's Theorem. This theorem, given by Torricelli in 1643, concerns the velocity of flow of a liquid from an orifice in a vessel. Consider a stream line starting from A and moving to B as shown in Fig. 9.13. At A the velocity of the stream line is zero if the height of the liquid is kept constant or approximately zero if the vessel has a large diameter. Also, the pressure p at both A and B is regarded as atmospheric. From Eqn. 9.17 Bernoulli's theorem for a stream line moving from A to B gives

$$\frac{p}{\rho} + gy_A = \frac{p}{\rho} + gy_B + \frac{V^2}{2}$$

where gy_A and gy_B are respectively the potential energy per unit mass at A and B, and V is the velocity of emergence of the liquid at B. Thus

$$V = \sqrt{2g(y_A - y_B)}$$
$$= \sqrt{2gh}$$

where h is the height of A above B. This result, known as Torricelli's theorem, was originally derived not by the above method but rather along the lines of reasoning given by Galileo for falling bodies.

Now the result given above is only approximately correct, for observation shows that the jet of liquid does not emerge from the orifice in a cylindrical form. The liquid coming out of the orifice is made up of a large number of stream lines converging towards the orifice. This convergence continues outside the orifice until there is a minimum cross section at some place C, Fig. 9.13, called the *vena contracta*. The ratio of the area of cross section at C to the area of the orifice is called the coefficient of contraction. This coefficient depends on the shape of the orifice. For a plane circular orifice the coefficient of contraction is experimentally found to be about 0.62.

Furthermore the convergence of the stream lines at the orifice implies that the pressure there is not atmospheric. At the vena contracta the stream lines are approximately parallel and the pressure there is equal to atmospheric. Hence the height h in Torricelli's theorem should be $(y_A - y_C)$ rather than $(y_A - y_B)$ as given above.

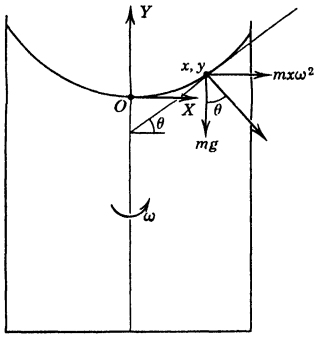


FIG. 9.14 The shape of the surface of a rotating liquid

9.8 The Shape of the Surface of a Rotating Liquid

When a liquid is set in rotation, as for example by stirring a cup of coffee with a spoon, the free surface is no longer horizontal but is parabolic in shape.

Consider a liquid in a cylindrical vessel rotating with a constant angular velocity ω about the Y axis as shown in Fig. 9.14. A fluid element of mass m at the point x, y on the surface of the liquid is in equilibrium under the action of its weight mg and the centrifugal reaction $mx\omega^2$ outward. If the tangent to the surface at x, y makes an angle θ with the horizontal, then from Fig. 9.14 we see that

$$\tan\theta = \frac{mx\omega^2}{mg} = \frac{x\omega^2}{g}$$

Since $\tan\theta = dy/dx$, we obtain the differential equation for the shape of the curve as

$$\frac{dy}{dx} = \frac{x\omega^2}{g}$$

By integration

$$y = \frac{x^2\omega^2}{2g} + C \tag{9.18}$$

If the X axis is drawn through the lowest point in the surface, then $x = 0$ at $y = 0$ and the constant of integration is zero. Thus the shape of the surface is parabolic.

Suppose now that the rotation of the cylindrical vessel is suddenly stopped with the liquid still rotating. Gradually the motion of the liquid stops, and as it does so we see that the fluid elements, near the stationary walls of the vessel, lag behind those near the axis. Thus there are tangential forces between contiguous fluid elements. These

tangential forces do not exist when the liquid is rotating as a solid body but only when there is a velocity gradient from the axis outward in the liquid. Up to the present we have assumed that we have been dealing with ideal fluids in which these tangential forces could not exist. We shall now investigate some of the properties of real liquids in connection with these tangential forces.

9.9 Viscosity

In order to simplify the understanding of these tangential forces we shall consider two plane horizontal plates with liquid placed between them. The lower plate is kept at rest while the upper plate, which is resting on the upper surface of the liquid, moves with a small constant velocity v. This is represented diagrammatically in Fig. 9.15.

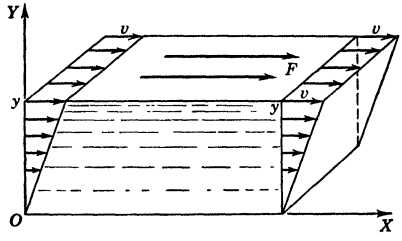


FIG. 9 15 Viscosity of a liquid.

We shall assume that the liquid in contact with the upper plate moves with this plate and that the liquid in contact with the lower plate is at rest. In other words there is no slippage between the plates and the liquid. If the velocity v of the upper plate is sufficiently small, then the liquid moves in horizontal stream lines or with laminar motion where each stream line moves relative to a contiguous one. The magnitude of the velocity v depends on the type of liquid, the distance between the plates, and the force F. If the velocity exceeds a critical amount, the regular laminar flow is broken up and turbulence occurs. We shall here assume that the velocity is below the critical velocity and that the liquid flows with this laminar motion.

In order for the motion to take place a tangential force F must be applied to the plates. Suppose that the plates have an area A and are separated by a vertical distance y. It is found experimentally that the force F required to produce laminar flow, with the upper layer moving with the velocity v, is proportional to the velocity v and to the area A and inversely proportional to the vertical distance y

between the plates. Thus

$$F \propto \frac{vA}{y}$$

The constant of proportionality in this equation is called the *coefficient of viscosity* η of the liquid. Hence

$$F = \eta \frac{vA}{y}$$

or

$$\eta = \frac{Fy}{Av} \tag{9.19}$$

The quantity F/A is the shearing stress, and the quantity v/y is called the velocity gradient.

The coefficient of viscosity of a liquid is the tangential force per unit area per unit velocity gradient. For two contiguous layers having a velocity gradient dv/dy the shearing stress F/A is given by

$$\frac{F}{A} = \eta \frac{dv}{dy} \tag{9.20}$$

From Eqn. 9.19 it follows that the dimensions of the coefficient of viscosity are $[MT^{-1}L^{-1}]$. The cgs unit of coefficient of viscosity is the dyne second per square centimeter which is also called a *poise*. It follows that *a coefficient of viscosity of 1 poise means that 1 dyne of force is required to maintain a tangential velocity difference of 1 cm per sec between two surfaces each a square centimeter in area and 1 cm apart.*

It is found experimentally that η decreases with temperature increase for liquids and increases with temperature increase for gases. A few values of η for liquids and gases are given below.

COEFFICIENTS OF VISCOSITY η

Substance	Temperature, °C	Coefficient of Viscosity, poises
Water	10	0 01308
Water	20	0.01005
Glycerin	0	121.1
Glycerin	20	14.9
Castor oil	10	24.2
Castor oil	20	9.86
Air	0	170.8×10^{-6}
Air	18	182.7×10^{-6}
Hydrogen	0	83.5×10^{-6}
Hydrogen	20	$87\ 5 \times 10^{-6}$

9.10 Measurement of the Coefficient of Viscosity by the Capillary Tube Method

About 1840 the French physicist Poiseuille carried out experiments on the flow of water through tubes of very small diameter. He found that the volume of water flowing per unit time through a horizontal capillary tube varied as the fourth power of the diameter of the tube, as the difference in pressure at the two ends, and inversely as the length of the tube. We shall now derive these experimental results from theoretical considerations.

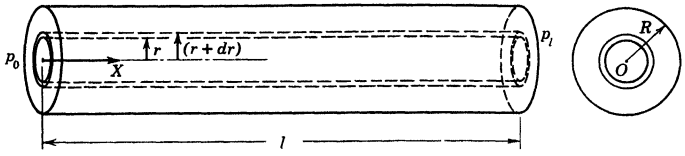


FIG. 9.16 The measurement of the viscosity of a liquid by a capillary tube.

Consider a horizontal capillary tube of length l and radius R, Fig. 9.16. Suppose that a liquid having a coefficient of viscosity η is moving with laminar flow in the tube. The velocity of the liquid is a maximum along the axis and is zero at the walls of the tube. Let the X axis lie along the axis of the tube, and let the pressure at $x = 0$ be p_0 and at $x = l$ be p_l.

Let us now consider the forces acting on an imaginary cylinder of radius r described about the axis of the tube. The radial velocity gradient at all points on the surface of this imaginary cylinder is dv/dr. Then from Eqn. 9.20 the force F exerted on the liquid inside the cylinder by the liquid outside is

$$F = 2\pi rl\eta \frac{dv}{dr}$$

where $2\pi rl$ is the area of the surface of the imaginary cylinder. The liquid inside the imaginary cylinder is moving with a larger velocity than that outside, and therefore the force F is to the left. This force is supplied by the pressure difference between the ends of the imaginary cylinder and is equal to

$$\pi r^2(p_0 - p_l)$$

Since the flow is steady, the forces acting on the liquid in the imaginary cylinder must be in equilibrium or

$$\pi r^2(p_0 - p_l) + 2\pi rl\eta \frac{dv}{dr} = 0$$

Thus

$$\frac{dv}{dr} = -\frac{r}{2\eta}\frac{(p_0 - p_l)}{l} \tag{9.21}$$

where $(p_0 - p_l)/l$ is the pressure gradient of the liquid in the tube. There is no radial pressure gradient and hence no radial flow since there is laminar flow of the liquid. The radial velocity gradient dv/dr is negative as shown by Eqn. 9.21 and also by the fact that the velocity is a maximum along the axis and zero at the walls. By integration of Eqn. 9.21, using the boundary conditions $v = 0$ at $r = R$, we have for the velocity v at any distance r from the axis

$$v = \frac{p_0 - p_l}{4\eta l}(R^2 - r^2) \tag{9.22}$$

In order to obtain the quantity of liquid flowing through the tube per unit of time we shall first consider a thin cylindrical shell whose radius is r and whose thickness is dr. The area of cross section of this shell is $2\pi r\, dr$, and the velocity of flow at the distance r is given by Eqn. 9.22. Thus the volume passing this cross section per second is

$$dQ = 2\pi v r\, dr$$

and the volume of liquid passing any cross section of the tube per second is

$$Q = \int_0^R 2\pi \frac{p_0 - p_l}{4\eta l}(R^2 r - r^3)\, dr$$

$$= \frac{\pi(p_0 - p_l)}{8\eta l} R^4 \tag{9.23}$$

This is known as Poiseuille's formula and agrees with his experimental results. From Eqn. 9.23 the coefficient of viscosity of a liquid can be obtained experimentally.

Suppose that the pressure difference of the liquid between the ends of an inclined capillary tube is maintained by connecting one end to a constant level tank at a height h above the entrance end of the capillary tube. The top of the liquid in the tank is at atmospheric pressure P_0, and its velocity may be taken as zero. If the velocity of the liquid in the center of the capillary is v, then the pressure p at the entrance end of the capillary is given by Bernoulli's equation as

$$P_0 + g\rho h = p + \frac{\rho v^2}{2}$$

The velocity v is obtained from Eqn. 9.22 by setting r equal to zero. At the exit end of the capillary the pressure is atmospheric, P_0, and,

if this exit end is at a vertical distance h_1 below the entrance end, the pressure difference $(p_0 - p_l)$ in Eqn. 9.23 becomes

$$p_0 - p_l = P_0 + g\rho h - \frac{\rho v^2}{2} - P_0 + g\rho h_1$$

$$= g\rho H - \frac{\rho v^2}{2}$$

where H is the vertical height $(h + h_1)$ of the exit end below the top of the constant level tank.

9.11 Measurement of the Coefficient of Viscosity by the Revolving Cylinder Method

In this method the liquid whose coefficient of viscosity is to be determined is placed between two concentric cylinders as shown in Fig. 9.17. The outer cylinder is rotated with a constant angular velocity ω_0 while the inner is at rest supported by a torsion fiber. Owing to the viscosity of the liquid, a torque is transmitted by the liquid to the inner cylinder which is balanced by the torque in the torsion fiber.

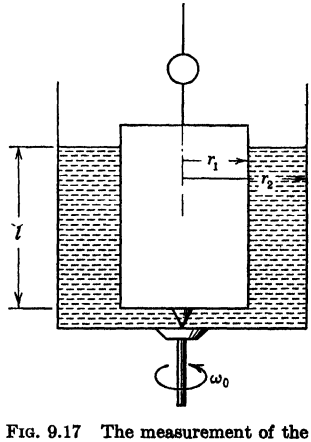


FIG. 9.17 The measurement of the viscosity of a liquid by a rotating cylinder.

In order to calculate the torque on the inner cylinder, consider an imaginary concentric cylinder of radius r and length l drawn within the liquid, where l is the length of the inner cylinder immersed in the liquid. If ω is the angular velocity of the surface of this cylinder of radius r, then the radial velocity gradient at this radius in the liquid is $r\, d\omega/dr$. At the outer cylinder where $r = r_2$ the angular velocity of the liquid is ω_0, and at the inner cylinder where $r = r_1$ the angular velocity of the liquid is zero. The liquid on the outer surface of this imaginary cylinder exerts a tangential force and a torque on the liquid on the inner surface of the cylinder. Since the surface area of the imaginary cylinder is $2\pi rl$, the total tangential force F exerted on this surface is given by Eqn. 9.20 as

$$F = 2\pi rl\eta r \frac{d\omega}{dr}$$

The torque L exerted by this tangential force is

$$L = Fr = 2\pi l\eta r^3 \frac{d\omega}{dr}$$

Since the liquid between the imaginary cylinder of radius r and the stationary cylinder of radius r_1 is in steady motion or undergoing no accelcration, the torque L is exerted on the stationary cylinder and is equal to that exerted by the imaginary cylinder.

By integrating the above equation, applying the boundary conditions of $\omega = 0$ at $r = r_1$ and $\omega = \omega_0$ at $r = r_2$, we have

$$\int_0^{\omega_0} 2\pi l\eta d\omega = \int_{r_1}^{r_2} L \frac{dr}{r^3}$$

or

$$2\pi l\eta\omega_0 = \frac{L}{2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

Now the torque L exerted on the inner cylinder is balanced by the torque in the supporting fiber. If the fiber has a torsion constant L_0 and is twisted through an angle of θ radians

$$L = L_0\theta$$

and

$$\eta = \frac{L_0\theta}{4\pi l\omega_0}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \tag{9.24}$$

In the above derivation we have neglected the torque exerted by the liquid on the under surface of the stationary cylinder. If there were regular stream-line motion between the two horizontal surfaces, it would be possible to calculate an expression for it. However, at the edge of the inner cylinder the stream lines would not be concentric with those nearer the axis. We can overcome this difficulty by making a determination of the angle of torsion θ for two different heights of the liquid. We must of course use the same angular velocity of rotation of the outer cylinder in both experiments.

The viscosity of liquids increases very much as their temperature is decreased. We are all familiar with this phenomenon since we change the oil in our car from one having a high viscosity in summer to one having a relatively low viscosity in winter. The grade number of car oil is proportional to the number of seconds a given quantity of oil takes to go through a capillary tube. The smaller the number, the less viscous or the "thinner" the oil.

9.12 The Velocity of Fall of a Sphere through a Viscous Liquid

This problem was first solved by Sir George Stokes about the middle of the last century. Since the method he used is beyond the scope of this book, we shall give only a partial solution by dimensional analysis. As was mentioned earlier, both sides of an equation between physical quantities must have the same dimensions. This statement is the foundation of dimensional analysis. In applying dimensional analysis to this problem, we must first attempt to ascertain the physical quantities that determine the force exerted on a sphere moving with constant velocity in a viscous liquid. This must be done by intelligent guesswork or intuition.

Suppose that a sphere moves through a viscous liquid with a constant small velocity so that there is stream-line motion. The applied force must be exactly equal and opposite to the friction force exerted by the liquid on the sphere, since we have assumed constant velocity. This frictional force F might be expected to depend on the size or radius r of the sphere, the viscosity η of the liquid, and the velocity v with which the sphere moves. If these are considered to be the only quantities on which the force depends, then we may say that the friction force is a function of the radius r, the viscosity η, and the velocity v. We can then write

$$F = Kr^a\eta^b v^c$$

where K is a dimensionless constant and a, b, c are dimensionless numbers to which r, η, and v must be raised respectively for the result to have the same dimensions as that of a force.

Writing out the dimensions of these quantities in terms of mass $[M]$, length $[L]$, and time $[T]$, we have

$$[MLT^{-2}] = [L]^a[ML^{-1}T^{-1}]^b[LT^{-1}]^c$$

Equating the powers for $[M]$, $[L]$, and $[T]$ respectively, we have:

$$\text{For } [M]: \qquad 1 = b$$

$$\text{For } [L]: \qquad 1 = a - b + c$$

$$\text{For } [T]: \qquad -2 = -b - c$$

The solution of these three equations gives

$$a = b = c = 1$$

Thus

$$F = Kr\eta v$$

The dimensionless constant K could be obtained by experiment by

actually measuring the force required to pull a sphere of known radius through a liquid of known viscosity at a known speed. If this were done accurately, then a value of 6π should be obtained. This is the value obtained theoretically by Stokes, and the expression

$$F = 6\pi r\eta v \tag{9.25}$$

is known as Stokes' law.

If we now return to the problem of a small sphere falling at a constant velocity through a viscous liquid, the effective weight of the sphere in the liquid is balanced by the frictional force. This effective weight is given by Archimedes' principle as

$$\tfrac{4}{3}\pi r^3(\rho_s - \rho_l)g$$

where r is the radius of the sphere of density ρ_s, and ρ_l is the density of the liquid. Thus we have for the velocity of fall v, using Eqn. 9.25,

$$F = 6\pi r\eta v = \tfrac{4}{3}\pi r^3(\rho_s - \rho_l)g$$

Hence

$$v = \frac{2g}{9\eta} r^2 (\rho_s - \rho_l) \tag{9.26}$$

From this equation one can experimentally determine the viscosity of a liquid since all the remaining quantities in Eqn. 9.26 can be easily measured. However, one must be careful to see that the motion is stream-line or laminar. This type of motion depends very much on the viscosity of the liquid and the shape of the body. A marble falling through glycerin does so with stream-line flow, and the motion is in a straight line with constant speed. If, however, the same marble is dropped through water, then the motion may be irregular because turbulence has been created in the liquid. Turbulent motion is very difficult to analyze, and we shall again have to revert to the method of dimensional analysis.

9.13 Turbulent Motion and Reynolds' Number

Toward the end of the last century Sir Osborne Reynolds investigated the change from laminar to turbulent flow in liquids in cylindrical tubes as the velocity of flow was increased. He found that for any liquid there was a critical velocity at which there was a sudden change from the laminar to the turbulent type of motion. His experiment consisted in inserting a narrow thread of colored liquid into the main body of liquid flowing through the tube. With laminar flow the narrow thread followed a straight line, but once the critical velocity was passed the thread broke up and the colored liquid ultimately diffused into the main body of the liquid. In order to determine the

point at which turbulence sets in we must ask ourselves on what quantities we might expect this critical velocity v_c to depend. These would seem to be the viscosity η and the density ρ of the liquid and the diameter d of the cylindrical tube into which the liquid is flowing. If these are the correct and only factors on which the critical velocity depends, we may write for dimensional purposes

$$v_c = R\eta^a\rho^b d^c$$

where R is a dimensionless number called Reynolds' number. Writing the dimensions of these quantities we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a[ML^{-3}]^b[L]^c$$

Then:

For $[L]$: $\quad 1 = -a - 3b + c$

For $[M]$: $\quad 0 = a + b$

For $[T]$: $\quad -1 = -a$

Hence from these equations we have

$$a = 1$$

$$b = -1$$

$$c = -1$$

Thus we have for the critical velocity v_c

$$v_c = R\frac{\eta}{\rho d} \tag{9.27}$$

Reynolds' number R is then given by

$$R = \frac{\rho d v_c}{\eta}$$

If the critical velocity is determined experimentally for a given liquid and diameter of pipe, then Reynolds' number can be found for a cylindrical pipe. Of course this analysis is only approximate for, as we have seen, the velocity of a liquid in a cylindrical tube varies from a maximum along the axis to zero at the edges of the tube when there is laminar flow. The critical velocity is the average velocity over the cross section of the tube.

Now when an object is pulled through a liquid so that there is laminar motion, then, as we know, the force varies with the velocity, but as soon as the motion becomes turbulent the force varies with the

square of the velocity. Eddies are set up behind the moving body so that the pressure behind the moving body is less than that in front, Fig. 9.18. Evidence of these eddies is seen when a car moves at a high speed over light objects that are whirled around in the air behind the car.

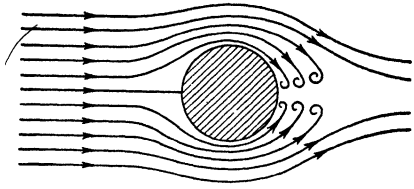


FIG. 9.18 Turbulent motion at an obstacle.

With laminar or stream-line motion no eddies are set up and there is no pressure difference between the front and back of the body moving through the fluid. In this the resistance force is entirely due to viscosity whereas with turbulence the resistance force is largely due to the pressure difference and is only slightly due to the viscosity.

The force due to pressure difference in the turbulent motion depends on the area of cross section A of the body perpendicular to the direction of the motion, on the velocity v of the body relative to that of the fluid, and on the density ρ of the fluid, which is either a gas or a liquid. Thus we may write the pressure resistance force F for dimensional purposes as

$$F = KA^a v^b \rho^c$$

where K is a dimensionless quantity. Writing out the dimensions of these quantities we have

$$[MLT^{-2}] = [L^2]^a [LT^{-1}]^b [ML^{-3}]^c$$

Equating the powers of $[M]$, $[L]$, and $[T]$ respectively, it follows that

$$\text{For } [M]: \qquad 1 = c$$

$$\text{For } [L]: \qquad 1 = 2a + b - 3c$$

$$\text{For } [T]: \qquad -2 = -b$$

Solving these equations, we have

$$a = 1 \qquad b = 2 \qquad c = 1$$

Hence

$$F = KA\rho v^2 \tag{9.28}$$

The value of the dimensionless constant K in Eqn. 9.28 depends on the shape of the body which is moving relative to the fluid. For any particular body it can be measured experimentally. It is about 0.7 for a flat disk and about 0.03 for a pear-shaped body.

9.14 Surface Tension

You have all probably noticed that when mercury is spilled over a table it collects into small drops and does not spread out into a thin film as water does. This difference in behavior between mercury and water is explained in terms of the surface tension of the two liquids. Another difference in the behavior of these two liquids that can also be explained by their surface tensions is shown by the fact that water rises in a capillary tube whereas mercury in a capillary tube is depressed below the outside level. That the surface of a stationary liquid behaves as though it were a stretched film is readily seen when a needle is made to rest on a water surface. This is, of course, contrary to Archimedes' principle since the needle is denser than the water. The needle lies in a small depression in the water, and if the surface is disturbed the needle immediately sinks to the bottom of the vessel containing the water.

This apparent surface film is considerably changed if a small piece of camphor is placed on it, for when the camphor dissolves in water the surface tension of the solution is much smaller than that of pure water. A small amount of oil on the surface of water also considerably changes the surface tension of the water. There are a number of so-called detergents which can so change surface tension relations between a duck and the water on which it floats that the duck will actually sink.

An elementary and simple explanation of surface tension can be given in terms of intermolecular forces. In the molecular theory of liquids it is assumed that the molecules of the liquid exert attractive forces over distances of about ten molecular diameters. These attractive forces decrease rapidly with distance from the molecular centers. Thus at an approximate distance of about 10^{-7} cm from the center of a molecule the attractive force becomes zero. This is, of course, only an approximate statement, for molecules of a liquid are not small stationary spheres like minute marbles. However, if we adopt this assumption of attractive forces between molecules, we shall be able to give a qualitative explanation of the phenomena associated with surface tension.

A molecule in the interior of a liquid or at a distance of a few molecular diameters from the surface is completely surrounded by other

molecules so that the resultant force on it is zero. This is not so for a molecule at the surface of a liquid. For such a molecule the upward force of attraction due to the molecules of the gas above the dense liquid is much less than the downward force due to the molecules of the liquid. There is, then, a resultant downward force on the molecules at the surface of a liquid. If a molecule is moved from the interior of the liquid to the surface, work must be done against this downward force. Thus the molecules at the surface possess greater potential energy than those in the interior of the liquid. It is this downward force on, or the larger potential energy of, the surface molecules that gives rise to the surface property called surface tension.

Since any system tends to attain a configuration in which its potential energy is a minimum, we might expect a liquid to assume the shape having the minimum surface area. For a given volume of liquid a sphere is that shape which has the minimum surface area. Thus when surface tension is the controlling factor we should expect liquids to have a spherical shape. This is what we find when small drops of liquid fall from the end of a fairly narrow vertical tube. Lead shot is made by allowing molten lead to fall through small circular holes at the top of a long cooling tower. As the size of the drops increases, so does their weight, and the potential energy due to gravity modifies the spherical shape. Small drops of mercury resting on a table are approximately spherical; larger ones tend to become flat on top.

9.15 Measurement of Surface Tension

The mechanical tension in the surface of a liquid may be simply demonstrated by the Dupré frame shown in Fig. 9.19. In this a clean rectangular wire frame with a freely sliding arm AB has a film of soap solution stretched over it. If the downward force F, which includes

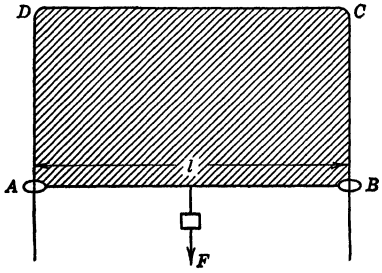


FIG. 9.19 Measurement of surface tension by a Dupré frame.

the weight of the wire AB, is sufficiently small, the wire will move upward due to the upward surface tension force of the soap film on the wire AB. Suppose that the system is in equilibrium with the force F. Then, since the film has two surfaces, the surface tension S is given by

$$S = \frac{F}{2l} \tag{9.29}$$

where l is the length of the wire AB. If the length l of the sliding arm is 5 cm and the force F is 250 dynes, the surface tension S of the soap film is 25 dynes per cm.

Suppose that the sliding arm AB, 5 cm in length, is pulled downward a distance of 6 cm so that the total area of the two sides of the soap film is increased by 60 cm^2. If the temperature remains constant, the work done by the external force is the product of the surface tension and the length of the arm AB, that is, 1500 ergs. Thus the work done per square centimeter when the temperature is constant is 1500 ergs/60 cm^2 or 25 ergs per cm^2. It follows that the surface tension may be expressed either as a force per centimeter or as the increase in surface potential energy per square centimeter.

When the film is stretched, molecules come to the new surface from the interior of the previously unstretched film. These new surface molecules have greater potential energy than those in the interior, and this energy comes at the expense of the kinetic energy of the molecules. If no heat were added to the system, the film would be cooled on stretching, or, conversely, heat must be absorbed by the film if its expansion is to be isothermal. In order to calculate the heat absorbed at constant temperature we must apply thermodynamic principles.

9.16 Carnot's Cycle and Surface Tension

Let us consider the surface of a liquid being extended and contracted under isothermal and adiabatic conditions. At constant temperature the surface tension is a constant independent of the area of the surface of the liquid. The initial surface tension and area of the surface is given by the point A in Fig. 9.20. Suppose that the area of the surface is increased from A to B by an amount a at a constant absolute temperature of T. If S is the surface tension at temperature T, the work done on the surface in expanding a cm^2 is Sa ergs. During this process an amount of heat of Q calories is absorbed by the surface in order to maintain the temperature constant at T.

Now let us expand the surface adiabatically from B to C as shown

in Fig. 9.20 so that the temperature falls to a temperature $(T - \Delta T)$. At this lower temperature let the surface tension be $(S + \Delta S)$. Continue this cycle by the isothermal CD and the adiabatic DA so that the surface is in the same condition after the cycle is completed as it was initially. Along the isothermal CD the surface is decreased in area a cm^2 and the work done by the surface is $(S + \Delta S)a$ ergs. While this decrease in area is taking place, an amount of heat $(Q + \Delta Q)$ is given out if the temperature is to remain constant.

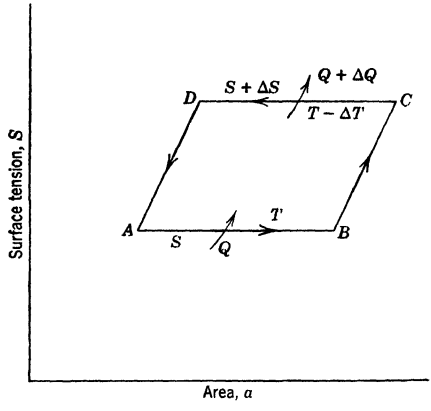


FIG. 9.20 Carnot's cycle and surface tension

In this cycle a quantity of heat Q has been absorbed by the surface at the temperature T, and an amount of heat $(Q + \Delta Q)$ has been given out at the temperature $(T - \Delta T)$. The net amount of work done by the surface is equal to the area of the figure $ABCD$ which is approximately a parallelogram whose area is

$$(S + \Delta S)a - Sa = a\,\Delta S$$

By the first law of thermodynamics the net amount of heat given out ΔQ must be equal to the work on the surface or

$$\Delta Q = -\frac{a\,\Delta S}{J}$$

where J is the value of the mechanical equivalent of heat or 4.2×10^7 ergs per calorie.

This Carnot cycle is reversible, and the second law of thermodynamics applied to it gives

$$\frac{\Delta Q}{Q} = \frac{\Delta T}{T}$$

Substituting for ΔQ gives

$$-\frac{a\,\Delta S}{JQ} = \frac{\Delta T}{T}$$

or

$$\frac{\Delta S}{\Delta T} = -\frac{JQ}{aT} \tag{9.30}$$

Now Q/a is the quantity of heat absorbed by each square centimeter increase in area at the constant absolute temperature T. This may be written $(dQ/da)_T$. Similarly $\Delta S/\Delta T$ is the rate of change of the surface tension with temperature when the area is constant and may be written $(dS/dT)_a$. Hence Eqn. 9.30 may be written

$$\left(\frac{dQ}{da}\right)_T = -\frac{T}{J}\left(\frac{dS}{dT}\right)_a \tag{9.31}$$

The surface tension S of water at 0°C is 75.60 dynes per cm, and the change in surface tension per degree centigrade at 0°C is approximately -0.14 dyne per cm per °C. Thus $(dQ/da)_T$, the heat absorbed at 0°C for each square centimeter increase in area of a water surface, is given by Eqn. 9.31 as

$$\left(\frac{dQ}{da}\right)_T = 273 \times 0.14$$

$$= 38.2 \text{ ergs per cm}^2$$

If, then, the surface of water at 0°C is increased by 1 cm^2, 75.60 ergs of work must be done by the external stretching agent and 38.2 ergs of work are absorbed by the surface in the form of heat. Experiments have shown that in many cases the heat required to maintain the temperature constant when a liquid surface is extended is approximately equal to half the work done in stretching the surface.

9.17 Angle of Contact of Liquids with Glass and Rise of Liquids in Capillary Tubes

You must have noticed the marked difference in the curvature of the free surfaces of water and mercury placed in small glass tubes. Water forms a concave surface relative to the air whereas mercury has a convex surface. Pure water wets a clean glass surface in such a manner that the angle of contact between the water and glass is zero,

as shown in Fig. 9.21*a*. With mercury the free surface makes an angle α with the glass surface at the point of contact. This angle of

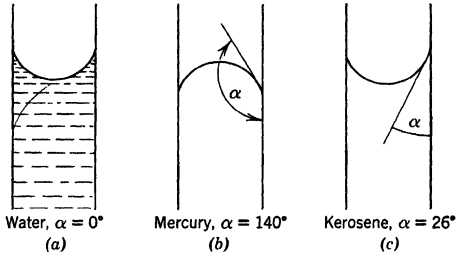


FIG. 9.21 The angle of contact for various liquids and glass surfaces.

contact α shown in Fig. 9.21*b* is of the order of 140° but changes somewhat after the mercury is exposed to air. For kerosene the angle of contact α is about 26° as shown in Fig. 9.21*c*.

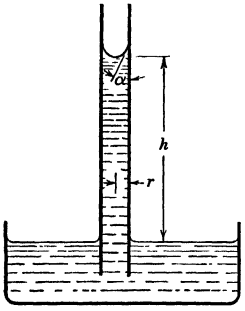


FIG. 9.22 The rise of a liquid in a capillary tube.

Let us now consider the rise of a liquid of density ρ, angle of contact α, and surface tension S, when placed in a uniform capillary tube of radius r, Fig. 9.22. The upward force on this liquid is $2\pi rS \cos \alpha$. This balances the weight of liquid in the tube. This liquid column has a height h measured to the lowest point in the meniscus. The weight of this cylindrical portion is $\pi r^2 h\rho g$. The small portion of liquid in the meniscus has a volume that is approximately the difference between a portion of a cylinder of height r and a hemisphere of radius r. This volume is therefore

$$\pi r^3 - \frac{2}{3}\pi r^3 = \frac{\pi r^3}{3}$$

Therefore the total weight of liquid in the column is

$$\pi r^2 h\rho g + \frac{\pi r^3}{3}\rho g$$

Hence

$$2\pi rS \cos \alpha = \pi r^2 h\rho g + \frac{\pi r^3}{3}\rho g$$

The surface tension S is then given by

$$S = \frac{g\rho r}{2 \cos \alpha}\left(h + \frac{r}{3}\right) \tag{9.32}$$

If the vapor of the liquid has a density ρ_0, then a portion of the weight of the liquid is balanced hydrostatically by the vapor and Eqn. 9.32 should be corrected to

$$S = \frac{gr(\rho - \rho_0)}{2 \cos \alpha}\left(h + \frac{r}{3}\right) \tag{9.33}$$

Actually the correction factor $r/3$ is valid only for tubes that are less than about 0.2 mm in diameter. For tubes of larger diameter other correction factors must be used which can be found in the literature.* This method of determining surface tension is capable of considerable accuracy if sufficient precautions are taken. This involves more complicated apparatus than that shown in Fig. 9.22 as can be appreciated by referring to the original literature. The value found by this method for the surface tension of water at 20°C is 72.785 dynes per cm. Though the capillary height method provides the highest accuracy, there are several other methods for the determination of surface tension. These methods depend on the weight of a drop falling from the end of a tube, or the force required to raise a circular ring of wire from the surface of a liquid. However, a rigorous discussion of the theory of these methods would be too complicated to be given here.

9.18 Excess Pressure inside a Spherical Bubble

As you may recall from your younger experiences with soap bubbles you blew into a pipe in order to form the bubbles. In other words the pressure inside the bubble must be greater than the outside atmospheric pressure. In order to calculate this excess of pressure let us imagine that the spherical bubble is cut along a diameter into two hemispheres, Fig. 9.23. The downward force on the upper half, considering the inner and outer surfaces of the bubble, is

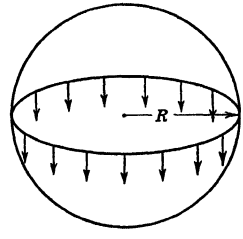


FIG. 9.23 Surface tension of a soap bubble.

$$2(2\pi RS)$$

* W. D. Harkins, "Determination of Surface and Interfacial Tension," in *Physical Methods of Organic Chemistry*, Interscience Publishers, New York, 1945, Vol. I, Chapter VI, p. 149.

where R is the radius of the bubble. This force along the surface balances the force due to the pressure over the cut surface whose area is πR^2. If p is the excess pressure inside the bubble over that outside, then for equilibrium

$$\pi R^2 p = 4\pi RS$$

or

$$p = \frac{4S}{R} \tag{9.34}$$

This result can also be applied to any spherical liquid surface. The pressure on the water side of the concave meniscus that occurs when water rises in a capillary tube is less than atmospheric. Similarly the pressure in the under side of a convex mercury meniscus in a capillary tube is greater than atmospheric.

PROBLEMS

1. A rectangular box has a base 3 ft long and 5 ft wide and a height of 6 ft. It is filled to the top with water having a density of 62.5 lb per ft^3. Find the resultant force on the bottom and on the vertical sides due to the water, and also the points where these forces act.

2. The top of a vertical gate is 40 ft below the level of the water at a dam. If the gate is 6 ft wide and 2 ft high and is hinged along its upper horizontal edge, find the torque exerted by the water about the line of hinges as an axis.

3. A horizontal cylindrical pipe 3 ft in radius has its ends closed with vertical slabs of glass Find the resultant force and the distance of this force from the center of the pipe when the pipe is half-filled with water.

4. The cross section of a wooden trough is an equilateral triangle, 2 ft on a side, with the upper side horizontal. Assuming that such a trough will hold water, find the force on one of the ends when the trough is filled with water.

5. A uniform solid sphere floats in water so that 0.8 of a vertical diameter is submerged in the water. Find the specific gravity of the sphere.

6. A hollow spherical iron shell is so made that it just floats when completely submerged in water. If the iron has a specific gravity of 7.8 and the shell has an outer diameter of 2 ft, find the inner diameter.

7. A U tube has a uniform diameter of 2 cm and contains 100 cm^3 of oil whose density is 0.8 gm per cm^3. The oil originally stands at the same horizontal level in the two vertical arms. By blowing into one of the arms the column of oil is set into oscillation with an amplitude of 1 cm. Assuming no damping, find the period of the simple harmonic vibrations of the oil and the force originally exerted by blowing on the oil.

8. A cylindrical steel tube, 6 in. in outside diameter and 2 ft long, is cut along a diameter. The vertical ends of the cut tube are closed by steel plates of the same thickness as the tube so as to form a boat. When placed in water it is found to float with its horizontal edges 1 in. above the horizontal level of the water. If the specific gravity of the steel is 7.8 find:

a. The thickness of the steel in the tube.

b. The center of gravity of this steel boat.

c. The center of buoyancy of this boat.

d. Whether you consider the boat to be stable or unstable.

e. The maximum volume of water the boat could contain without sinking.

9. Show that for any point in a liquid at rest having a density ρ and a pressure p, the force per unit mass in the X direction is

$$F_x = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

10. If p_0 is the pressure at sea level and p is the pressure at the top of a column of air h meters in height and having a uniform temperature of T degrees absolute, show that

$$\log_{10} p_0 - \log_{10} p_1 = 0.0148\,\frac{h}{T}$$

Take the density of air at 0°C as 0.001293 gm per cm^3 and the pressure p_0 of air at sea level as 1.01×10^6 dynes per cm^2.

11. Water flows out of a circular hole in the side of a large cylindrical tank 1 meter in diameter. Initially the water level is 4 meters above the vena contracta which has a circular cross section of 2-cm radius. Find the volume of water escaping from the tank per second when the water level is 3 meters above the vena contracta, taking into account the velocity of fall of the top of the liquid in the tank. Find how long it takes to empty the vessel.

12. At a fixed point the velocity of the water in a stream line is 40 cm per sec and the rate of change of the velocity with distance along the stream line is 5 cm per sec per cm. Find the acceleration of the liquid at the fixed point.

13. Write out the dimensions of each term in Eqn. 9.15.

14. From Bernoulli's principle, explain why a ping-pong ball can stay within a vertical air current issuing from a tube.

15. Liquid is flowing in a horizontal pipe that has a constriction along its length. Vertical tubes are attached at both the wide and the constricted portions to serve as manometers. Suppose that there is a stopcock at the exit end. Would the liquid in the manometer tubes rise or fall on closing the stopcock? Explain your answer.

16. The cross section of the wide portion of the pipe in question 15 is 10 $in.^2$ and at the constricted portion is 2.5 $in.^2$ Water is flowing at the rate of 2 ft per sec in the wide portion of the pipe where a pressure gage reads 4 lb per $in.^2$ Find the reading of a pressure gage inserted in the constricted portion and the number of cubic feet of water flowing through the pipe per minute.

17. Find the volume of water flowing per minute through a horizontal pipe 6 in. in diameter if there is a difference in height of 2 ft in the levels of the liquid in the two manometers, one of which is placed at the wide portion and the other at the constricted portion. The constricted portion has a diameter of 2 in.

18. A horizontal tube 1 mm in diameter and 20 cm long is connected at one end to a constant level water tank whose height is 100 cm. If 578 cm^3 of water flows through the tube in 8 min, find the coefficient of viscosity of water.

19. A horizontal capillary tube whose radius is R and whose length is l is connected to an airtight vessel whose volume is V. Enough air escapes from the vessel through the capillary tube during a time of t seconds to reduce the pressure of the air in the vessel from P_1 to P_2 without any change in temperature. If atmospheric presure is P_0, prove that the coefficient of viscosity η of air is given by

$$\eta = \frac{\pi R^4 P_0 t}{8lV \log_e \dfrac{(P_2 + P_0)(P_1 - P_0)}{(P_2 - P_0)(P_1 + P_0)}}$$

(In order to prove this relationship consider a small section of the capillary tube of length dx where the pressure is p and then integrate over the length l of the tube. If P is the pressure at the entrance to the capillary and P_0 is atmospheric pressure at the end of the tube, then show that the time rate dV/dt at which air enters the capillary is

$$P\frac{dV}{dt} = \frac{\pi R^4}{16\eta l}(P^2 - P_0^2)$$

From Boyle's law

$$P\frac{dV}{dt} = -V\frac{dP}{dt}$$

where V is the constant volume of the vessel and dP/dt is the rate of change of pressure in the vessel.)

20. A capillary tube 0.04 cm in radius and 100 cm in length is connected to a vessel having a volume of 4000 cm^3. The original pressure of the air in the vessel is 80.5 cm of mercury, and after 20 sec the pressure has fallen to 79.4 cm of mercury. If the temperature of the air is constant throughout the experiment and atmospheric pressure is 76 cm of mercury, find the coefficient of viscosity of air.

21. A liquid having a coefficient of viscosity η is flowing through a vertical capillary tube whose length is l and whose radius is R. If the exit end of the capillary is a vertical height H below a constant level tank, then show that the volume of liquid Q emerging from the capillary per second may be given as

$$Q = \frac{\pi R^4 g\rho}{8\eta l}\left(H - \frac{2Q^2}{g\pi^2 R^4}\right)$$

22. A pitot tube mounted on the leading edge of an airplane indicates a pressure difference of 0.01 atm on a differential air barometer. What is the velocity of the airplane?

23. Assuming that Reynolds' number is 1000 for cylindrical pipes, calculate the critical velocity of water and of glycerin at 20°C in tubes of 5-cm diameter. The density of glycerin is 1.36 gm per cm.^{3}

24. In the determination of the coefficient of viscosity by the rotating cylinder method as described in section 9.11, two heights of liquid l_1 and l_2 are used and the corresponding angles of torsion θ_1 and θ_2 are observed. If the radii of the inner and outer cylinders are r_1 and r_2 respectively, and Ω is the angular velocity of rotation of the outer cylinder in both cases, and L_0 is the torsion constant of the suspension, show that according to Eqn. 9.24 the coefficient of viscosity η of the liquid is

$$\eta = \frac{L_0(\theta_1 - \theta_2)}{4\pi\Omega(l_1 - l_2)}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

25. The coefficient of viscosity η of a gas or a liquid can be found by measuring the torsion $L_0\theta$ exerted on a horizontal circular plate placed a small distance d above a concentric and parallel plate which is rotating with an angular velocity Ω.

If R is the radius of the circular plates, show that

$$\eta = \frac{2dL_0\theta}{\pi R^4 \Omega}$$

26. Assuming that the maximum diameter of raindrops depends on their surface tension, density, and the acceleration due to gravity g, find, from dimensional analysis, the order of magnitude of the mass of the largest drops.

27. A soap film having a thickness of 10^{-4} cm and a temperature of 0°C is stretched adiabatically until its area is doubled. Assuming that the specific heat and density of the soap solution are each equal to unity, find the fall in temperature of the film if its surface tension decreases at the rate of 0.15 dyne per cm per °C rise in temperature.

28. Find the weight of a drop of water that might be expected to fall from the plane end of a glass tube having a diameter of 0.5 cm. The surface tension of water is 72.8 dynes per cm. Find the force due to surface tension that would be needed to raise a circular ring whose diameter is 2 cm from the surface of water. (Both results will be only approximate as the conditions are not so simple as assumed.)

29. From the fact that the pressure on the underside of the meniscus of water in a capillary tube is less than atmospheric, find the height to which water rises in a capillary tube of radius r.

30. Four small spherical drops of water, each of 1-mm radius, coalesce to form a single drop. Show that the surface potential energy of the four drops is about 1.59 times the total surface potential energy of the single coalesced drop.

10 · WAVE MOTION

10.1 Wave and Particle Motion

Classical physics has two main branches: the one deals with the motion of particles or rigid bodies, and the other with wave motion. These two branches developed independently up to about the beginning of the present century. Until then electrical charges had, along with many other concepts, been explained in terms of particles, whereas light and sound phenomena were explained in terms of wave theory. Very briefly we may say that the motion of particles can be interpreted by Newton's laws, together with the additional developments of the eighteenth and nineteenth centuries made by Laplace, Lagrange, Hamilton, and others. According to the particle theory energy is carried by the moving particle, whereas the energy of a spherical wave spreads out in all directions. The course of physics in the present century has shown that natural phenomena cannot be explained quite so simply. Light or electromagnetic radiation has both wave and particle properties. The properties of electrons are also interpreted in terms of both particles and waves. These two distinct theories, each formerly used independently to explain certain natural phenomena, have, in a sense, been merged into the modern theory of wave mechanics.

In this chapter we shall be concerned with some of the fundamental characteristics of the propagation of waves in continuous media such as solids, liquids, and gases. When an oscillatory motion is impressed on some part of a body, this motion is propagated as a wave motion to all parts of the body. Up to the present our only discussion of vibratory motion impressed on a body has been concerned with the oscillations of a spring. We may properly ask whether waves were not propagated in the spring? Actually waves were set up in the spring, but they were so small that they could be neglected, since we were dealing with springs whose mass was negligible compared to the mass of the body attached to its end. When the mass of a spring is assumed to be zero, no inertial force is required to set any portion of the spring in motion, so that any force applied at one end of the spring is passed on to the other end undiminished by each elementary portion

of the spring. This force remains the same throughout the spring, and every elementary length of the spring is stretched an amount proportional to its length, so that the resultant extension of the whole spring is proportional to the applied force.

Let us drop the assumption that the spring has zero mass. Now a force is required to accelerate each elementary portion of the spring. The force at any instant is no longer the same throughout the spring, and waves are propagated in the spring. Wave motion occurs in any medium in which mass and restoring force are associated with each elementary portion of the medium. In our previous discussion of springs we considered the restoring force as evenly distributed throughout the spring whereas the mass undergoing acceleration was considered to be concentrated at the lower end of the spring. A similar situation occurs in electrical circuits. So long as we assume that inductance and capacitance are separate or lumped parameters we deal with forced oscillations or alternating currents and not with true waves. Although in practice there is always some capacitance associated with an inductance and some inductance with a capacitance, these small effects can generally be neglected if the frequency of the alternations is small. When the frequency is large these effects cannot be neglected and waves are propagated. Here again the parameters, inductance, and capacitance, are distributed throughout the circuit.

For most practical examples the medium in which the wave is propagated is limited and not infinite. At the boundaries of the medium the waves are reflected so that the medium is acted on by both the incident and the reflected waves. With suitable geometrical configurations, standing waves can be set up in the medium. These may occur in a string fastened at both ends, in a membrane such as a drumhead clamped around its edges, and so forth. Before discussing the phenomenon of standing waves let us find an expression for the velocity of propagation of transverse waves in an infinitely long cord.

10.2 Transverse Waves in a Cord

When a stretched cord is caused to vibrate either by giving sinusoidal vibrations to one end or by plucking it, transverse waves travel along the cord. In this transverse wave the motion of the elementary portions of the cord is at right angles or transverse to the direction of the traveling wave. The wave is a motion of a geometrical pattern transmitting energy and not a mass motion of the cord as a whole.

In order to determine the velocity of a wave in a cord let us consider a uniform cord whose mass per unit length is μ. The cord is subjected to a constant tension F. We shall assume that the deflections of the

cord are so small that the tension remains constant throughout the deflections and that damping can be neglected. Suppose that at some instant of time the cord assumes the configuration shown in Fig. 10.1*a*. An element of the cord whose horizontal length is Δx at position x, y is shown enlarged in Fig. 10.1*b*. On the ends of this element the forces make angles θ_1, θ_2 respectively with the horizontal X axis. The problem is to find the motion of the element AB under the action of

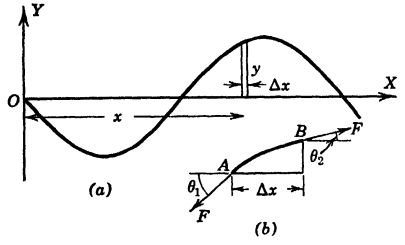


FIG. 10.1 Transverse wave in a cord.

these forces. We shall assume that the displacements are so small that $\sin\theta \approx \tan\theta$, where the notation $\approx$ means approximately equal to. The mass of the element AB may be approximately taken as $\mu\,\Delta x$ since $\cos\theta \approx 1$. The resultant vertical force on the element is $F(\sin\theta_2 - \sin\theta_1)$, and this may be set equal to $F(\tan\theta_2 - \tan\theta_1)$. Now

$$F\tan\theta_1 = F\left(\frac{\partial y}{\partial x}\right)_A \tag{10.1}$$

where $(\partial y/\partial x)_A$ is the slope of the tangent at A at the instant of time under consideration. It is necessary to use partial derivatives since y is a function of both the position x and the time t. The resultant vertical force on the element of length Δx is then

$$F\left(\frac{\partial y}{\partial x}\right)_B - F\left(\frac{\partial y}{\partial x}\right)_A$$

By Taylor's theorem, using only the first-order terms,

$$\left(\frac{\partial y}{\partial x}\right)_B = \left(\frac{\partial y}{\partial x}\right)_A + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

The resultant vertical force F_y on an element of length Δx is

$$F_y = F\frac{\partial^2 y}{\partial x^2}\Delta x \tag{10.2}$$

and by Newton's second law this is equal to the mass of the element multiplied by the vertical acceleration. Hence

$$F\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{10.3}$$

This is the wave equation for waves propagated along the X axis. You should be able to show from the assumptions that the resultant horizontal force is zero. From Eqn. 10.3 it follows that $\sqrt{F/\mu}$ has

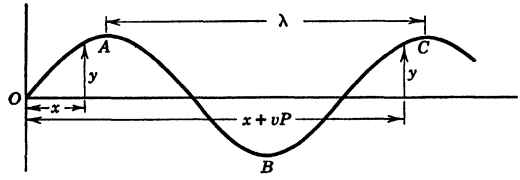


FIG. 10.2 Equal displacements on a transverse wave a wave length $\lambda = vP$ apart.

the dimensions of a velocity which we shall see is the velocity of the transverse waves in the cord. The wave equation above is a linear homogeneous partial differential equation of the second order. There are several methods of obtaining a solution of such an equation. Because of its importance we shall illustrate two of these methods. It may be readily verified by substitution that

$$y = f(x - vt) \tag{10.4}$$

is a solution of the wave equation. In this equation f is any function—sine, cosine, square, cube, etc. By substituting Eqn. 10.4 in Eqn. 10.3 it is easily shown that $v = \sqrt{F/\mu}$. Similarly $y = f(x + vt)$ is also a solution. Let us ask what $y = f(x - vt)$ signifies physically. It represents a wave traveling in the positive X direction, and $y = f(x + vt)$ represents a wave traveling in the negative X direction. Suppose that a wave is traveling along the positive X direction, Fig. 10.2. If there is a displacement y at position x and time t, then, if the wave is traveling in the positive X direction, there is the same displacement y at some position $(x + vP)$ to the right of x at the time $(t + P)$ or a time P later. Mathematically this is given as

$$\begin{aligned} y &= f(x - vt) \\ &= f(x + vP - vt - vP) \end{aligned}$$

An important special case of the function f is the sine or cosine function. Consider the solution

$$y = A \sin k(x - vt) \tag{10.5}$$

where A is the maximum displacement and is called the *amplitude.* It gives the value of the height of the crest at A or C or the depth of the trough at B. The quantity $k(x - vt)$ is called the phase of the vibration, and the distance between two points in the same phase, such as AC, is called the *wave length* λ. If P is the *period* of the wave, then, if the wave is traveling to the right, there is the same displacement at x and t as there is at $(x + \lambda)$ and $(t + P)$. Hence

$$A \sin k(x - vt) = A \sin k[x + \lambda - v(t + P)]$$

or, since the arguments of the sine function are equal,

$$\lambda = vP \qquad v = \frac{\lambda}{P} \tag{10.6}$$

The quantity v is the ratio of the distance λ traveled by the wave to the period P. It is called the *phase velocity* of the wave since it gives the velocity at which any point of any phase travels along. The *frequency* f of the waves is $1/P$ as indicated in the discussion of simple harmonic motion. This phase velocity may be understood as follows: If a given value is taken for y, then for this to be constant the quantity $(x - vt)$ must also be constant. The derivative of this latter quantity with respect to time gives $v = dx/dt$. This is the velocity of a point of constant phase and is therefore called the phase velocity of the wave.

It may be seen that every element of the cord executes simple harmonic motion with a period of P and a frequency of f. For any particular point where x is a constant

$$y = A \sin (k_1 t + k_2)$$

where k_1 and k_2 are constants depending on the period P and the position x.

The quantity k in Eqn. 10.5 may be obtained in terms of the wave length by noting that the distance λ corresponds to an increase in the argument of the sine function of 2π radians:

$$k(x - vt) + 2\pi = k(x + \lambda - vt)$$

or

$$k = \frac{2\pi}{\lambda} \tag{10.7}$$

Thus the displacement y at any point x and time t on the rope along

which a simple harmonic wave is traveling in the positive X direction may be written

$$\begin{aligned} y &= A \sin k(x - vt) \\ &= A \sin \frac{2\pi}{\lambda}(x - vt) \\ &= A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{P}\right) \\ &= A \sin (kx - \omega t) \\ &= A \sin \omega\left(\frac{x}{v} - t\right) \end{aligned} \tag{10.8}$$

where $\omega = 2\pi/P = 2\pi v/\lambda = kv$.

Since the wave equation is linear and homogeneous, it follows that

$$y = f_1(x - vt) + f_2(x + vt)$$

is also a solution, as may be readily verified by substitution. Now this solution deals with waves traveling in opposite directions along the string. If these waves are simple harmonic and have the same amplitude, they give rise to what are called standing waves. The equation of such standing waves is given by

$$\begin{aligned} y &= A \sin (kx - \omega t) + A \sin (kx + \omega t) \\ &= 2A \sin kx \cos \omega t \end{aligned} \tag{10.9}$$

and this satisfies the wave equation, Eqn. 10.3. These standing waves will be discussed in a later section.

10.3 Solution of the Wave Equation by Separation of the Variables

The one-dimensional form of the wave equation for a transverse wave in a cord of mass per unit length μ and tension F placed along the X axis is given by Eqn. 10.3 as

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

In this partial differential equation the dependent variable is the displacement y. It is a function of the two independent variables x, the position along the cord, and t, the time under consideration. A relatively simple method of solving such a partial differential equation is to find a solution that is the product of functions of each of the

independent variables. Let us assume that a product solution is possible of the type

$$y = X(x)T(t)$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone. From this equation it follows by differentiation that

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2} \qquad \text{and} \qquad \frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2}$$

The wave equation becomes

$$T\frac{d^2X}{dx^2} = \frac{\mu X}{F}\frac{d^2T}{dt^2}$$

Dividing by XT gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{\mu}{F}\frac{1}{T}\frac{d^2T}{dt^2}$$

In this equation the left-hand side is a function of x alone and the right-hand side of t alone. These two sides cannot be equal unless each is equal to some constant, k'. Thus we have

$$\frac{d^2X}{dx^2} = k'X \qquad \text{and} \qquad \frac{d^2T}{dt^2} = k'T\frac{F}{\mu}$$

These two equations are homogeneous differential equations of the second order. We have encountered such equations in the discussion of simple harmonic motion where k' was intrinsically negative. Since any point x on the cord is undergoing simple harmonic motion, we are led to believe that k' must be negative in this problem. To emphasize the negative character of k' we shall put $k' = -c^2$. The equations resulting from the separation of the variables then become

$$\frac{d^2X}{dx^2} = -c^2X \qquad \text{and} \qquad \frac{d^2T}{dt^2} = -\frac{F}{\mu}c^2T$$

From the procedure outlined in Chapter 5 the solutions of these equations are

$$X = A\cos(cx + \alpha) \qquad \text{and} \qquad T = B\cos\left(\sqrt{\frac{F}{\mu}}ct + \beta\right)$$

where A, B, α, and β are constants. It would have been equally valid to use the sine function in place of the cosine. The displacement y of any point x on the cord at a time t is given by

$$y = XT = AB\cos(cx + \alpha)\cos\left(\sqrt{\frac{F}{\mu}}ct + \beta\right)$$

By a trigonometric transformation this may be written

$$y = \frac{AB}{2}\cos\left(cx + \sqrt{\frac{F}{\mu}}ct + \alpha + \beta\right) + \frac{AB}{2}\cos\left(cx - \sqrt{\frac{F}{\mu}}ct + \alpha - \beta\right)$$

In order to interpret this equation we shall set $\sqrt{F/\mu}$ equal to the phase velocity v of the wave and the quantity c equal to $2\pi/\lambda$. The latter follows since cx and vct must be angles so that c must have the dimensions of an inverse length. This length is the wave length λ of the wave in the cord. Thus

$$y = \frac{AB}{2}\cos\left(2\pi\frac{x}{\lambda} + \frac{2\pi vt}{\lambda} + \alpha + \beta\right) + \frac{AB}{2}\cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi vt}{\lambda} + \alpha - \beta\right)$$

$$= \frac{AB}{2}\cos 2\pi\left(\frac{x}{\lambda} + \frac{t}{P} + \frac{\alpha+\beta}{2\pi}\right) + \frac{AB}{2}\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{P} + \frac{\alpha-\beta}{2\pi}\right)$$

where $P = \lambda/v$ is the period of the waves. The solution represents two progressive waves: the one traveling in the negative X direction and the other in the positive X direction. Their amplitudes are $AB/2$, and their phases at $x = 0$, $t = 0$ are respectively $(\alpha + \beta)/2\pi$ and $(\alpha - \beta)/2\pi$. These quantities are determined from the initial or boundary conditions of the particular problem.

From the above we see that the wave equation, Eqn. 10.3, is satisfied by any of the above solutions representing a wave traveling along the positive X axis or similar expressions for a wave traveling along the negative X axis.

10.4 Energy in a Progressive Wave

We are all familiar with the fact that waves transmit energy since many of the important aspects of waves are associated with this transmission of energy. Let us calculate the energy per unit length of the cord, Fig. 10.1, in which there is a wave progressing to the right whose configuration is described by

$$y = A \sin k(x - vt) \tag{10.8}$$

This is the equation of a plane-polarized plane wave traveling along the positive X axis with a velocity v. It is confined or polarized in the XY plane and is a plane wave since its direction of propagation is along the X axis perpendicular to the YZ plane. In our problem the velocity is determined by the tension and mass per unit length of the cord whereas the frequency of the waves may have any value. At any particular point in the string the energy is changing from kinetic to potential and back again so that to obtain the energy of

the wave per unit length it is necessary to calculate both the kinetic and potential energies. The kinetic energy of a unit length of the cord is $\mu\dot{y}^2/2$, where μ is the mass per unit length. For any particular point the vertical velocity is, from Eqn. 10.8,

$$\dot{y} = -Akv \cos k(x - vt)$$

and the K.E. per unit length is

$$\frac{A^2\mu}{2} k^2v^2 \cos^2 k(x - vt)$$

The potential energy is the negative of the work done by the restoring force in putting the rope into any particular configuration. For any point the restoring force per unit length is given by Eqn. 10.2 as

$$F_y = F\frac{\partial^2 y}{\partial x^2} = -FAk^2 \sin k(x - vt) = -Fk^2y$$

and the potential energy per unit length is

$$V = -\int_0^y F_y\, dy = Fk^2\frac{y^2}{2} = \frac{Fk^2}{2} A^2 \sin^2 k(x - vt)$$
$$= \frac{A^2}{2} \mu k^2v^2 \sin^2 k(x - vt)$$

since $v^2 = F/\mu$. The average values of the kinetic energy and the potential energy per unit length are each equal to $A^2\mu k^2v^2/4$ and the total energy per unit length is the constant

$$E_{\text{total}} = \frac{A^2\mu k^2v^2}{2} = \frac{A^2\mu\omega^2}{2} \tag{10.10}$$

which is independent of the time.

10.5 Longitudinal Waves in a Solid Rod

A *longitudinal* wave is one in which the displacement of any part of the medium is along the direction of propagation of the wave. Sound waves are longitudinal waves in air. A perfect fluid—that is, a perfect gas or a perfect liquid—can transmit longitudinal waves but not transverse waves since a perfect fluid has no shear modulus.

As an introduction to longitudinal waves we shall first determine the velocity of propagation of a longitudinal wave in a uniform rectangular rod. Figure 10.3 shows such a rod of a uniform cross section A and a density ρ oriented along the X axis. This rod is subjected to a longitudinal force which varies with position and time as

would occur if a longitudinal wave were passing through the rod. This varying force causes any cross section of the rod to undergo small longitudinal displacements about a mean position. It is these varying displacements that constitute the wave motion. We shall assume that the displacements are within the elastic limit of the material and that any damping can be neglected.

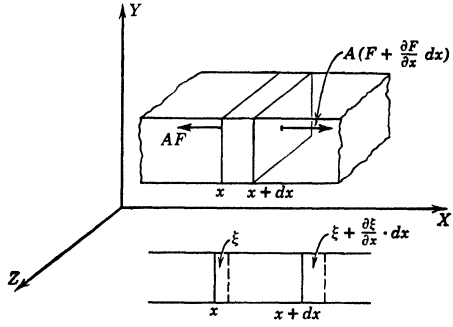


FIG. 10.3 Analysis of a longitudinal wave in a solid rod.

Let us now consider a thin slab of this rod at position x and of a thickness dx. At some particular instant of time let the force per unit area transmitted to the slab by the material of the rod at the left of the slab be F so that the total force is FA. Since the force varies with x, its value at $(x + dx)$ may be expressed to a sufficient degree of accuracy by the first term in Taylor's expansion as

$$A\left(F + \frac{\partial F}{\partial x} dx\right)$$

This is the force exerted on the slab by the material to the right of it. The resultant force acting on the slab is

$$A \frac{\partial F}{\partial x} dx$$

Suppose that ξ is the displacement from its equilibrium position of the cross section in the X direction. This is the displacement of the slab that has a mass $\rho A\, dx$. From Newton's second law the acceleration $\partial^2 \xi / \partial t^2$ is given as

$$\rho A\, dx \frac{\partial^2 \xi}{\partial t^2} = A \frac{\partial F}{\partial x} dx$$

or

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial F}{\partial x} \tag{10.11}$$

Since the stress on the rod is different at different points and times, it follows that the strain must likewise vary with position and time. At position x the displacement is ξ, and at $(x + dx)$ the displacement is $\xi + (\partial \xi / \partial x)dx$. The strain on the slab is the resultant displacement $(\partial \xi / \partial x)dx$ divided by the original length dx, or the strain is $\partial \xi / \partial x$. From the definition of Young's modulus E for a rod

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{\partial \xi / \partial x}$$

Hence

$$F = E \frac{\partial \xi}{\partial x}$$

and

$$\frac{\partial F}{\partial x} = E \frac{\partial^2 \xi}{\partial x^2}$$

Thus from Eqn. 10.11 the equation of motion of the slab becomes

$$\rho \frac{\partial^2 \xi}{\partial t^2} = E \frac{\partial^2 \xi}{\partial x^2} \tag{10.12}$$

This is the wave equation for a longitudinal wave traveling in the rod with a phase velocity v of

$$v = \sqrt{\frac{E}{\rho}}$$

This longitudinal wave is a plane wave inasmuch as the displacement ξ is the same at all points on any given cross section.

The energy per unit volume of the waves in the rod may be calculated in a similar manner to that used for the waves in a cord. If a harmonic wave of amplitude A given by

$$\xi = A \sin (kx - \omega t)$$

is traveling through the rod, the average energy per unit volume E_v is

$$E_v = \frac{A^2 \omega^2 \rho}{2} \tag{10.13}$$

where ω is the angular velocity of the wave and equals $2\pi v/\lambda$. This energy travels through the rod with the phase velocity v since it is the wave that transmits the energy.

Another concept associated with waves is that of the intensity of a wave. This is defined as the amount of energy passing per second through each unit area of the medium perpendicular to the direction of propagation of the wave. For the plane harmonic wave in the rod the intensity I is

$$I = \frac{A^2\omega^2\rho v}{2} \tag{10.14}$$

Intensities of waves are generally expressed in ergs per square centimeter per second or in watts per square meter. An average person can just hear a sound having an intensity of 0.000001 microwatts per square meter.

10.6 Longitudinal Waves in a Gas

The analysis of the motion of plane waves in a gas is almost the same as that for the rod discussed in the previous section. Suppose that the plane wave is traveling through the gas in the X direction. If the

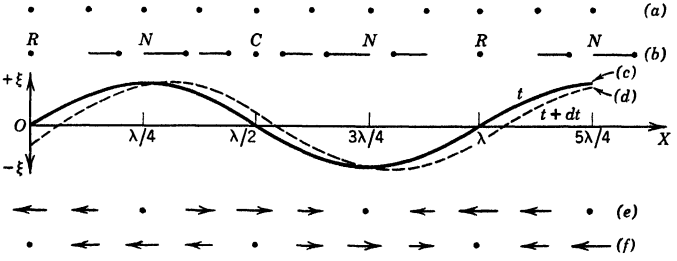


FIG. 10 4 Longitudinal wave in a gas. (*a*) Equilibrium positions. (*b*) Displaced positions. (*c*) Longitudinal displacements replaced by equivalent vertical displacements at time t; (*d*) at time $t + dt$. (*e*) Velocity of particles at time t. (*f*) Accelerations of particles at time t.

gas is considered to be composed of molecules, then as the wave passes through the gas the molecules vibrate about a mean position along the X axis. This motion is superimposed on the random motion of the molecules. All molecules in the YZ plane having a given value of x on the average have the same displacement at any instant, or the plane wave front of the wave is in the YZ plane. The displacement of the molecules about their mean positions produces rarefactions and compressions in the gas. These are shown in Fig. 10.4, where R indicates the position of a rarefaction, N the position of normal pressure, and C the position of a compression. In Fig. 10.4*a* are shown the

normal position of the molecules; and diagram b shows their position when displaced any amount ξ at some instant of time. This displacement varies with x, the position of the molecule. Where the molecules crowd together there is a *compression*, and where they are far apart there is a *rarefaction*. Notice that at the center of a rarefaction or compression there is no displacement, whereas normal pressure occurs at maximum displacements. In Fig. 10.4*c* the longitudinal wave is represented as a transverse wave by rotating the displacements through 90°, thus making displacements to the right positive and vertically up, and displacements to the left negative and vertically down. According to this convention $\partial\xi/\partial x$ has a maximum negative value at a compression, a maximum positive value at a rarefaction and a value of zero at a position of normal pressure.

Suppose that the wave is traveling to the right. After a short time dt, the wave has the configuration shown by the dotted curve (d). From this the velocities of the molecules may be obtained. The displacement velocity and acceleration of the molecules can be also obtained from the equation of the wave

$$\xi = A \sin \frac{2\pi}{\lambda} (x - vt)$$

The curve c is drawn for $t = 0$, so for this time

$$\xi = A \sin \frac{2\pi x}{\lambda}$$

The velocity of the particles at any time t and position x is

$$\frac{\partial \xi}{\partial t} = -Av \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (x - vt)$$

which for $t = 0$ becomes

$$\frac{\partial \xi}{\partial t} = -Av \frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda}$$

This is shown in curve e. Similarly the accelerations are given by $\partial^2\xi/\partial t^2$, which at time $t = 0$ is

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{4\pi^2}{\lambda^2} Av^2 \sin \frac{2\pi x}{\lambda}$$

These are shown in diagram f.

Let us now return to the determination of the velocity of sound in a gas. Consider a thin section of the gas of thickness dx located

at x as shown in Fig. 10.3. At some instant of time the displacement of the particles at x is ξ and at $(x + dx)$ is

$$\xi + \left(\frac{\partial \xi}{\partial x}\right) dx$$

If ρ is the density of the gas in the thin section, then the motion of the gas is given by Eqn. 10.11 as

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial F}{\partial x}$$

The modulus of elasticity involved is the bulk modulus k_m instead of Young's modulus E as for a solid. The increase in volume of the slab due to the different displacements of its two faces is

$$A \left(\frac{\partial \xi}{\partial x}\right) dx$$

where A is the cross section of the slab. The original volume of the slab is $A\,dx$ so that the volume strain (dV/V) is

$$\frac{\partial \xi}{\partial x}$$

Now the bulk modulus of the gas k_m is given by

$$k_m = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{(\partial \xi / \partial x)}$$

where F is the stress on the thin slab. Hence

$$F = k_m \frac{\partial \xi}{\partial x}$$

The equation of motion becomes

$$\rho \frac{\partial^2 \xi}{\partial t^2} = k_m \frac{\partial^2 \xi}{\partial x^2}$$

This is a wave equation, and the velocity of the progressive waves is given by

$$v = \sqrt{\frac{k_m}{\rho}}$$

The question now arises as to the value of the bulk modulus k_m for the gas. If the temperature remains constant as the wave passes through it, then the pressure and volume are related by Boyle's law, or the pressure multiplied by the volume is constant.

If PV = a constant, then $P\,dV + V\,dP = 0$, or

$$k_m = \frac{\text{Stress}}{\text{Strain}} = \frac{dP}{-dV/V} = P$$

Notice that the increase in pressure or stress dP causes a decrease in volume dV or a strain of $-dV/V$. Thus, if the temperature remains constant, the velocity of longitudinal waves or the velocity of sound in a gas is given by

$$v = \sqrt{P/\rho}$$

This expression was first obtained by Newton, who realized that it was definitely lower than the experimental value but failed to correct it in an acceptable scientific manner. The method* Newton used to make the correction is interesting inasmuch as he knew the experimental value of the velocity of sound in air. It shows that when the answer is known even the best of scientists may make mistakes in theory!

It was Laplace who, in 1816, pointed out that when sound waves pass through a gas the condensations and rarefactions respectively heat up and cool down the gas. The compressions and rarefactions take place so rapidly that there is no appreciable heat loss and the condensations and rarefactions are adiabatic rather than isothermal. Thus the relation between pressure and volume of the gas through which a sound wave is passing is given by the adiabatic equation for a gas

$$PV^\gamma = \text{Constant}$$

where γ is the ratio of the specific heat of the gas at constant pressure to that at constant volume. For air, γ is approximately 1.41. It follows from Laplace's correction that the velocity of sound in a gas is given by

$$v = \sqrt{\frac{\gamma P}{\rho}} \tag{10.15}$$

The formula gives values that agree very well with the experimental values for the velocity of sound in a gas.

10.7 Waves in Unlimited Solids

Up to the present we have discussed the velocity of transverse waves in a string, and of longitudinal waves in a rod and in a gas. Although the mathematical analysis of other types of wave motion is

* *The Principia.* A revision of Motte's translation by F. Cajori, University of California Press, 1934, p. 382.

beyond the scope of this text, we should not leave this subject without giving some of the results. In an unlimited homogeneous isotropic medium the velocity of the longitudinal waves v_l is given by

$$v_l = \sqrt{\frac{k_m + 4n/3}{\rho}}$$

where k_m is the bulk modulus, n the shear modulus, and ρ the density of the medium. An informal derivation of this equation will be given because it is related to the earlier work on elasticity.

The velocity of longitudinal waves in a rod is given by $v = \sqrt{E/\rho}$ where E is Young's modulus. Now in the definition of Young's modulus we are concerned only with the longitudinal strain and not with the lateral strain which, according to Poisson's ratio, is σ times

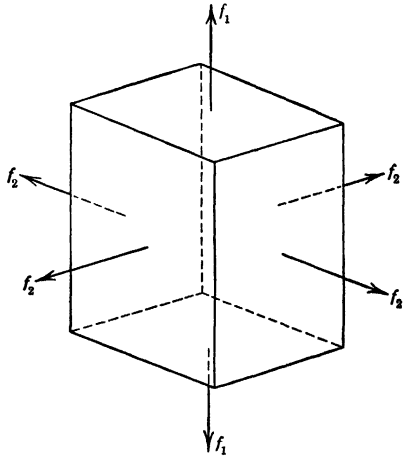


FIG. 10.5 Forces on a portion of an unlimited solid.

as great. The distinction between a rod and an unlimited homogeneous medium is the fact that when there is a longitudinal strain in the rod there is a lateral strain, whereas in the unlimited homogeneous medium no lateral strain is present. Thus we must be able to determine the longitudinal strain when there is no lateral strain. To do this let us consider a rectangular parallelopiped under the stresses f_1 and f_2 as shown in Fig. 10.5. The stresses f_2 are applied so that there is no change in length or strain in the directions of these stresses.

The problem is to determine the strain in the direction of f_1 when there is no strain in either of the directions perpendicular to f_1.

If all these stresses act simultaneously, then, following the argument in Chapter 7, the strain in the direction of f_1 is

$$a = \frac{f_1}{E} - 2\frac{f_2\sigma}{E}$$

In the direction of f_2 the strain must be zero or

$$\frac{f_2}{E} - \frac{f_2\sigma}{E} - \frac{f_1\sigma}{E} = 0$$

Hence

$$f_2(1 - \sigma) = f_1\sigma$$

Substituting for f_2 in the first of these equations gives for a the strain in the direction of f_1:

$$a = \frac{f_1}{E}\left(1 - \frac{2\sigma^2}{1 - \sigma}\right) = \frac{f_1}{E}\left(\frac{1 - \sigma - 2\sigma^2}{1 - \sigma}\right)$$

$$= \frac{f_1}{E}\frac{(1 + \sigma)(1 - 2\sigma)}{1 - \sigma}$$

The modulus of elongational elasticity is then given by

$$\frac{f_1}{a} = \frac{E(1 - \sigma)}{(1 + \sigma)(1 - 2\sigma)}$$

Using Eqns. 7.6 and 7.9, which are

$$E = 3k_m(1 - 2\sigma) \qquad \text{and} \qquad \sigma = \frac{3k_m - 2n}{6k_m + 2n}$$

it follows that

$$\frac{f_1}{a} = k_m + \frac{4n}{3}$$

It is left as an exercise for you to show that the velocity of any particular type of wave in a medium is given by the square root of the appropriate modulus of elasticity of the medium divided by its density. Thus the velocity of longitudinal waves in an unlimited homogeneous isotropic medium of density ρ is given by

$$v_l = \sqrt{\frac{f/a}{\rho}} = \sqrt{\frac{k_m + 4n/3}{\rho}} \tag{10.16}$$

Now we shall prove that the velocity of transverse waves in an

unlimited homogeneous isotropic medium is given by $v_t = \sqrt{n/\rho}$. To do this consider a thin slab of medium thickness dx with sides perpendicular to the XY plane, Fig. 10.6. Suppose that a plane transverse wave whose wave front is parallel to the sides of the slab passes through the medium. As a result of this a point A on the X axis at position x is moved vertically a distance of y to A', and a point B

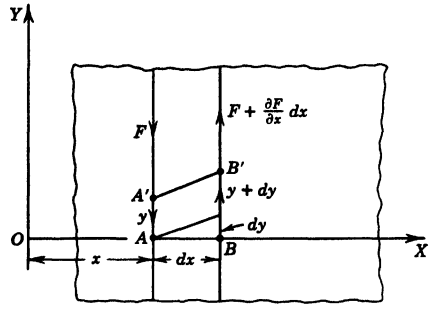


FIG. 10 6 Transverse forces on a portion of an unlimited solid.

on the other face at position $x + dx$, originally on the axis, is moved to B', a distance of $y + dy$ from its original position. There is a shearing action on this slab due to the wave. Suppose that the shearing stress on the face through AA' is F and that on the face through BB' is $F + (\partial F/\partial x)dx$. Then for a unit area of the slab, by Newton's second law,

$$\frac{\partial F}{\partial x}\,dx = \rho\,\frac{\partial^2 y}{\partial t^2}\,dx$$

From the definition of the shear modulus n it follows that

$$n = \frac{F}{\partial y/\partial x} \qquad \text{or} \qquad F = n\,\frac{\partial y}{\partial x}$$

Thus

$$n\,\frac{\partial^2 y}{\partial x^2} = \rho\,\frac{\partial^2 y}{\partial t^2}$$

which is the wave equation and gives for the velocity of a transverse wave through an elastic medium

$$v_t = \sqrt{\frac{n}{\rho}} \tag{10.17}$$

Thus we see that the velocity of longitudinal waves in an unlimited solid is larger than the velocity of the corresponding transverse waves. This is of importance in seismology where the waves created by earthquakes travel out in all directions. The first wave to reach a seismograph from an earthquake is the longitudinal one, and then follows the transverse one. If the time interval between the arrival of the longitudinal and that of the transverse waves is measured, then the distance of the earthquake from the seismograph can be determined. This, of course, involves a knowledge of the difference in velocity of the two types of waves. When the earthquake waves reach the surface, they create surface waves whose velocity we have not calculated. Since a perfect liquid does not possess any shear modulus, there is no transverse wave in the body of it but only a longitudinal wave whose velocity is $\sqrt{k/\rho}$. There are, of course, surface waves on a liquid as, for example those on the surface of a body of water. These waves may appear to be transverse but are not really so. The individual particles of the water move in ellipses.

When a wave strikes a boundary, it is partially reflected and partially transmitted, the relative amplitudes of the two waves depending on the densities of the media on the two sides of the boundary. We shall now proceed to calculate the relative amplitudes of waves on reflection.

10.8 Reflection of Waves at a Boundary

Since it is easier to visualize transverse waves than longitudinal ones, we shall consider the reflection and the transmission of a transverse wave at the boundary of two ropes of different linear densities.

Suppose that two ropes, having masses per unit length of μ_1, μ_2

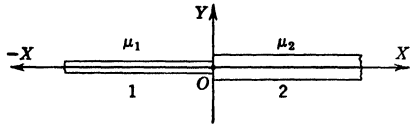


FIG. 10.7 Reflection of waves at a boundary.

respectively, lie along the X axis in their equilibrium position and are joined at the origin, $x = 0$, Fig. 10.7. The tension F in the two ropes is the same, and the velocity of the transverse wave in rope 1 is v_1 and in rope 2 is v_2. Let us investigate what happens to a wave traveling from the left toward the right when it arrives at the boundary. Any transmitted or reflected wave must have the same frequency f as the incident wave. For the incident wave in rope 1 the displacement

at any time is given by

$$y_i = A_i \sin 2\pi f\left(t - \frac{x}{v_1}\right)$$

and for the reflected wave in rope 1 traveling to the left by

$$y_r = A_r \sin 2\pi f\left(t + \frac{x}{v_1}\right)$$

where A_i and A_r are the amplitudes of the incident and reflected waves. The transmitted wave in rope 2 moving to the right is represented by

$$y_t = A_t \sin 2\pi f\left(t - \frac{x}{v_2}\right)$$

At the boundary, $x = 0$, the vertical displacement of the two ropes must be the same at every instant of time, or

$$y_i + y_r = y_t$$

From this it follows that

$$A_i + A_r = A_t$$

Also at the boundary, $x = 0$, the vertical forces on the ropes must be the same. From Eqn. 10.1 this gives

$$F\left(\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x}\right)_{x=0} = F\left(\frac{\partial y_t}{\partial x}\right)_{x=0}$$

By substitution in the above equations we have

$$-\frac{A_i}{v_1} + \frac{A_r}{v_1} = -\frac{A_t}{v_2}$$

or

$$A_i - A_r = \frac{v_1}{v_2} A_t$$

Substituting for A_t we have

$$\frac{A_i - A_r}{A_i + A_r} = \frac{v_1}{v_2} \tag{10.18}$$

If the coefficient of reflection R is defined as the ratio of the amplitude of the reflected wave to the incident wave, then

$$R = \frac{A_r}{A_i} = \frac{v_2 - v_1}{v_2 + v_1} \tag{10.19}$$

The velocities in the two ropes are

$$v_1 = \sqrt{\frac{F}{\mu_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{F}{\mu_2}}$$

It follows that

$$R = \frac{\dfrac{1}{\sqrt{\mu_2}} - \dfrac{1}{\sqrt{\mu_1}}}{\dfrac{1}{\sqrt{\mu_2}} + \dfrac{1}{\sqrt{\mu_1}}} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \tag{10.20}$$

If the mass per unit length μ_1 of rope 1 is smaller than μ_2 of rope 2, then the coefficient of reflection is negative and the amplitude of the reflected wave A_r is opposite to that of the incident wave. The reflected wave is 180° out of phase with the incident wave at the boundary. This is the case when a rope is tied to a wall or when a sound wave is reflected at a closed end of an organ pipe. The displacements of the air molecules in the incident and reflected waves are in opposite directions at the closed end. A compression is reflected as a compression and a rarefaction as a rarefaction.

On the other hand, if μ_1 is greater than μ_2, then R is positive. There will be no change in phase at reflection, so A_r has the same sign as A_i. In acoustics this corresponds to reflection at the open end of an organ pipe, and for this a compression is reflected as a rarefaction and vice versa.

The incident and reflected waves traveling in the same medium with the same velocity, frequency, and wave length can interfere with each other in such a manner as to produce standing waves. We shall next investigate the conditions necessary for setting up standing waves.

10.9 Standing Waves

Although progressive waves in various media have occupied much of our attention, it is perhaps well to remember that there are no such entities as infinite media. Sooner or later a wave meets a boundary and is reflected. The medium is then acted on by the two waves, the incident and the reflected. To investigate the action of these two waves we shall consider a cord fastened at both ends. If the cord is plucked or excited as by a violin bow, standing waves are set up along the cord.

The incident and reflected waves whose wave length is λ and whose period is P may be represented at any time by

$$y = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{P}\right) + A \sin 2\pi\left(\frac{x}{\lambda} + \frac{t}{P}\right)$$

where it is assumed that the amplitude A of the two waves is equal. By expanding the sines it follows that the resultant displacement y at any point x and time t is

$$y = 2A \sin 2\pi \frac{x}{\lambda} \cos 2\pi \frac{t}{P}$$

This is a solution of the wave equation, Eqn. 10.3.

Suppose that a cord of length l is rigidly clamped at $x = 0$ and $x = l$. At these two points the displacement y is zero at all times. The boundary condition, $y = 0$ at $x = 0$, is already satisfied by the equation. To satisfy the condition $y = 0$ at $x = l$ for all values of t we must have

$$\sin 2\pi \frac{l}{\lambda} = 0$$

or

$$2\pi \frac{l}{\lambda} = n\pi$$

where $n = 1, 2, 3 \cdots$. If λ_1 is the value of λ corresponding to $n = 1$, then

$$\lambda_1 = 2l$$

Similarly

$$\lambda_2 = \frac{2l}{2}, \qquad \lambda_3 = \frac{2l}{3}, \qquad \lambda_4 = \frac{2l}{4}, \qquad \lambda_n = \frac{2l}{n} \cdots$$

The corresponding frequencies are

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2l}, \qquad f_2 = \frac{2v}{2l}, \qquad f_3 = \frac{3v}{2l}, \qquad f_n = \frac{nv}{2l} \cdots$$

The lowest frequency f_1 is called the *fundamental*, and the other possible frequencies $f_2, f_3, \cdots, f_n$ are called the *harmonics* or *overtones*. For the nth possible or allowable vibration

$$y = 2A \sin \frac{n\pi x}{l} \cos 2\pi f_n t \tag{10.21}$$

This is the equation of a standing wave. There are some points, called *nodes*, at which $\sin (n\pi x/l)$ or y is always zero and therefore the cord is always at rest at these points. There are other points called the *antinodes* or *loops* where $\sin (n\pi x/l)$ equals unity. Such points oscillate with maximum amplitude. The nodes and loops for the fundamental and the first two harmonics are shown in Fig. 10.8.

It is instructive to write the equation for the standing waves in the equivalent trigonometric form representing two waves traveling in opposite directions. From these we may show the manner in which

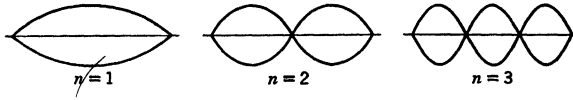


FIG. 10.8 The fundamental and first two harmonics in a standing wave.

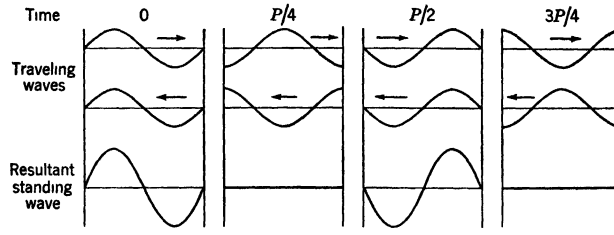


FIG. 10.9 Formation of standing waves by two similar waves traveling in opposite directions.

the standing wave is built up The nth vibration of the standing wave may be written, using $f_n = nv/2l$, as

$$y = 2A \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{l}$$

$$= A \sin \frac{n\pi}{l}(x - vt) + A \sin \frac{n\pi}{l}(x + vt)$$

In Fig. 10.9 the formation of standing waves is shown. These are produced by two similar waves traveling in opposite directions.

10.10 Vibration of a Cord with a Given Initial Displacement

As we have seen, a cord fixed at both ends is capable of vibrating with many different frequencies f_n related to each other in a simple manner. When such a cord is plucked, the initial displacement and velocity of each part of the cord may be quite complicated. However, the cord will vibrate with many of the frequencies f_n simultaneously, the fundamental usually having the largest amplitude whereas the amplitudes of the harmonics generally decrease considerably with the frequency.

Since the wave equation for a cord having a linear density μ and a tension F

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

is a linear homogeneous equation, the complete solution is the sum of the solutions for each frequency. If ϕ is the phase angle, the displacement y at any point x and time t is, from Eqn. 10.21,

$$y = \sum_{n=1}^{n=\infty} 2A_n \sin\frac{n\pi x}{l}\cos\left(\frac{n\pi vt}{l} - \phi\right)$$

since $f_n = nv/2l$, A_n is the amplitude of the nth harmonic wave, and Σ denotes the summation of the solutions. In a more general form the displacement at any point x and time t is given by

$$y = \sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l}\left(2A_n\cos\frac{n\pi vt}{l} + 2B_n\sin\frac{n\pi vt}{l}\right) \qquad (10.22)$$

Let us now consider a cord fixed at $x = 0$ and $x = l$ with its center point pulled out a distance h at zero time, Fig. 10.10. The problem

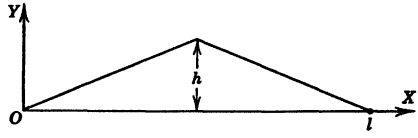


FIG. 10.10 The initial displacement of a cord.

is to determine the amplitudes A_n of the waves at the various frequencies that arise when the cord is released. At time $t = 0$ the displacement at x, from Eqn. 10.22, is

$$y_{(x,0)} = y_0 = \sum_{n=1}^{n=\infty} 2A_n \sin\frac{n\pi x}{l} \qquad (10.23)$$

and the initial velocity $\partial y/\partial t$ at $t = 0$ is

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = u_0 = \frac{2\pi v}{l}\sum_{n=1}^{n=\infty} nB_n \sin\frac{n\pi x}{l} \qquad (10.24)$$

A series such as those given by the right-hand side of Eqns. 10.23 and 10.24 is known as a Fourier series.

About 1700 the French physicist, Saveur, noted that a cord could vibrate with several harmonic frequencies at the same time. In 1755 Daniel Bernoulli showed mathematically that this was possible and that the resultant displacement at any point was the algebraic sum of the displacements due to each harmonic. At this time several of the great mathematicians, among them Euler, saw that Bernoulli's conclusion meant that any arbitrary function, such as that describing the initial shape of a vibrating cord as given above, could be expressed by an infinite series of sines and cosines, and this they seriously doubted. In 1822 a distinguished French military engineer and mathematician, Joseph Fourier, published a great work, *Théorie Analytique de la Chaleur*, in which he presented what has become known as the Fourier series. Fourier showed that an arbitrary function can with certain restrictions be expressed as a series of sines and cosines with appropriate coefficients. The Fourier series is of the greatest importance in every branch of physics, especially in sound, heat, and electricity. To develop this series in a complete manner would take us far afield, so we shall limit ourselves to the particular problem at hand. Mathematically no completely satisfactory proof has been given to show that it is possible to express any function satisfying the boundary conditions in terms of a Fourier series of the type $\sum_{n=1}^{n=\infty} A_n \sin (n\pi x/l)$. However, as physicists we may be satisfied with the experimental proofs showing that the initial shape and velocity of a cord does definitely specify the motion of the cord.

We shall now proceed with the problem of the cord set in motion by raising its center a distance h and then letting go. Since the initial velocity of the cord, $u_0(x)$, is zero, it follows that the coefficients B_n in Eqn. 10.24 in the velocity series must each be zero. The values of the coefficients A_n are not so easily disposed of; otherwise there would be no problem. To obtain these coefficients we use the mathematical trick of multiplying each side of Eqn 10.23 for $y(x, 0)$ by $\sin (\pi mx/l)\, dx$ and integrating from 0 to l. This procedure is useful because the integral

$$\int_0^l \sin \frac{\pi nx}{l} \sin \frac{\pi mx}{l}\, dx$$

is zero for every integral value of m except where $n = m$, at which point its value is $l/2$. Try it and see if this is not true! Thus

$$\int_0^l y_0 \sin \frac{\pi mx}{l}\, dx = \sum_{n=1}^{n=\infty} 2A_n \int_0^l \sin \frac{\pi nx}{l} \sin \frac{\pi mx}{l}\, dx$$

$$= lA_m$$

or

$$A_m = \frac{1}{l}\int_0^l y_0 \sin\frac{\pi m x}{l}\,dx \tag{10.25}$$

We shall break the integral into two parts, one from 0 to $l/2$ and the other from $l/2$ to l. In the interval from 0 to $l/2$ the equation of the initial configuration of the string is

$$\frac{y}{x} = \frac{h}{l/2} \qquad \text{or} \qquad y = 2\frac{hx}{l}$$

and in the region from $l/2$ to l the equation is

$$\frac{y}{l-x} = \frac{h}{l/2} \qquad \text{or} \qquad y = 2\frac{h(l-x)}{l}$$

Thus

$$A_m = \frac{1}{l}\left[\int_0^{l/2} \frac{2hx}{l}\sin\frac{\pi m x}{l}\,dx + \int_{l/2}^{l}\frac{2h}{l}(l-x)\sin\frac{\pi m x}{l}\,dx\right]$$

$$= \frac{4h}{\pi^2 m^2}\sin\frac{\pi m}{2} \tag{10.26}$$

This may be proved by integrating by parts. If m is an even integer, then the corresponding A_m's are zero. If m is an odd integer, then, recalling that $\sin \pi/2 = 1$ and $\sin 3\pi/2 = -1$, it follows from Eqn. 10.26 that

$$A_m = \frac{4h}{\pi^2 m^2}(-1)^{(m-1)/2} \tag{10.27}$$

From Eqn. 10.22

$$y = \sum_1^\infty 2A_m \sin\frac{m\pi x}{l}\cos\frac{m\pi v t}{l}$$

Then, using Eqn. 10.27, it follows that

$$y = \frac{8h}{\pi^2}\left(\sin\frac{\pi x}{l}\cos\frac{\pi v t}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi v t}{l}\right.$$
$$\left. + \frac{1}{25}\sin\frac{5\pi x}{l}\cos\frac{5\pi v t}{l} - \frac{1}{49}\sin\frac{7\pi x}{l}\cos\frac{7\pi v t}{l} + \cdots\right) \tag{10.28}$$

Using $l = 5$ and $h = 1$, the first four terms are plotted in Fig. 10.11 for $t = 0$. We see from the above equation that even harmonics are absent for a cord started in the manner described. These even harmonics would imply a node at the center which does not exist since the cord is initially displaced at the center. Since the intensity of a wave is proportional to the square of its amplitude, then for the sound

emitted by the string the fundamental would have an intensity 81 times the third harmonic and 625 times the fifth harmonic, etc.

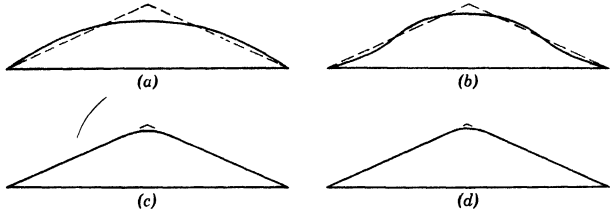


FIG. 10 11 The initial displacement of the cord given by the Fourier series, Eqn. 10 28: (*a*) one term; (*b*) two terms; (*c*) three terms; (*d*) four terms. The dotted curve shows the initial displacement.

In an organ pipe, wind is blown past a lip and eddies are set up which produce the fundamental and the overtones. To compute the relative intensities of these is a difficult matter since the initial conditions are difficult to determine.

10.11 Velocity of a Group of Waves. Group Velocity

In every example discussed so far the velocity of the waves has not depended on their wave length. All waves, whatever their wave length, traveled with the same velocity but this is not always so. In both surface water waves and in light waves traveling through some medium the velocity of the waves depends on their wave length and there is then dispersion. The effect of this dispersion is frequently seen when a motorboat moves rapidly through a body of calm water. An inverted V-shaped wave front is produced with the point of the V at the moving boat. Individual small waves of various wave lengths are produced by the boat, and these move out with their respective velocities and interfere in such a manner as to produce the inverted V-shaped wave front. We shall now proceed to calculate the velocity of a group of waves produced by interference of waves of different wave length traveling with different velocities through the medium.

Suppose that two harmonic waves of the same amplitude but having slightly different velocities, wave lengths, and frequencies are traveling along the positive X axis. If wave 1 of wave length λ, velocity v, and amplitude A produces a displacement y_1 of the medium at a point x and time t, then y_1 may be expressed

$$y_1 = A \sin \frac{2\pi}{\lambda} (x - vt)$$

Similarly, if wave 2 of wave length $(\lambda + d\lambda)$, velocity $(v + dv)$, and amplitude A produces a displacement y_2 of the medium at the same point x and time t, then

$$y_2 = A \sin \frac{2\pi}{\lambda + d\lambda} [x - (v + dv)t]$$

The resultant displacement y at the point x and time t is given by the principle of superposition of the waves as

$$y = y_1 + y_2 = A \sin \frac{2\pi}{\lambda} (x - vt) + A \sin \frac{2\pi}{\lambda + d\lambda} [x - (v + dv)t]$$

This expression can be transformed by using the trigonometric identity

$$\sin P + \sin Q = 2 \sin \frac{P + Q}{2} \cos \frac{P - Q}{2} = 2 \sin \frac{\text{sum}}{2} \cos \frac{\text{difference}}{2}$$

Hence

$$y = 2A \left\{ \sin \pi \left[\frac{x}{\lambda} + \frac{x}{\lambda + d\lambda} - \frac{vt}{\lambda} - \frac{(v + dv)t}{(\lambda + d\lambda)} \right] \cos \pi \left[\frac{x}{\lambda} - \frac{x}{\lambda + d\lambda} - \frac{vt}{\lambda} + \frac{(v + dv)t}{(\lambda + d\lambda)} \right] \right\}$$

Assuming that $d\lambda$ and dv are small compared to λ and v respectively, we have

$$y = 2A \sin \frac{2\pi}{\lambda} (x - vt) \cos \pi \left[x \frac{d\lambda}{\lambda^2} - \frac{(v\, d\lambda - \lambda\, dv)t}{\lambda^2} \right] \quad (10.29)$$

This complicated expression represents a wave traveling with a phase velocity v given by the sine term. The amplitude of the wave is changing as the wave travels along with the group velocity given by the cosine term. If Eqn. 10.29 is plotted for some instant of time, then curves similar to those shown in Fig. 10.12 are obtained. The full curve is the sine term representing a wave of wave length λ and velocity v whose amplitude changes in a manner shown by the dotted curve. This dotted curve is given by the cosine term. The points of maximum or minimum amplitude travel along the X axis with a velocity U called the group velocity, where

$$U = \frac{(v\, d\lambda - \lambda\, dv)/\lambda^2}{d\lambda/\lambda^2} = v - \lambda \frac{dv}{d\lambda} \quad (10.30)$$

If, as is the case for water waves, the larger wave lengths have the

larger velocity, that is, $dv/d\lambda$ is positive, then the group velocity U is smaller than the phase velocity v of the waves. The individual waves move from the origin to the front, creating the larger crests and troughs which travel with the slower velocity U. The group waves are produced by the interference of the smaller waves advancing through the group from the origin to the front of the group where they die out and are replaced by new waves from the origin.

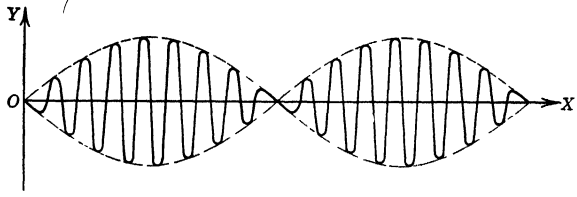


FIG. 10.12 Group velocity curves. The dotted line wave curve, the cosine curve in Eqn. 10 29, moves with the group velocity U, and the full curve, the sine term, moves with the phase velocity v.

In optics the concept of group velocity played an important role in reconciling the results obtained for the ratio of the velocity of light in air and in carbon disulfide by direct measurements with those obtained by refractive index methods. It has also been of importance in founding the modern theory of wave mechanics.

An interesting analogy to the bow wave produced by a fast moving boat is that of the electromagnetic radiation given off by a charged particle moving in a medium with a speed greater than that of light in the medium. This radiation is called Čerenkov radiation.* It can be observed with electrons moving with almost the speed of 3×10^{10} cm per sec in lucite in which the speed of light is about 2×10^{10} cm per sec.

10.12 Dimensional Analysis of the Velocity of Sound in a Gas

In order to illustrate the method of dimensional analysis we shall consider the problem of the velocity of sound in a gas. Let us assume that at any given temperature the velocity of sound v depends on the density ρ, the pressure p, and the viscosity η of the gas. Then the velocity v may be expressed

$$v = K\rho^a p^b \eta^c \tag{10.31}$$

where K is a dimensionless constant and a, b, c are dimensionless

* D. Halliday, *Introductory Nuclear Physics*. John Wiley and Sons, New York, 1950, p. 191.

numbers which must be determined. Let us now put in the dimensions of the quantities in Eqn. 10.31 in terms of mass, length, and time. We have:

$$[LT^{-1}] = [ML^{-3}]^a[ML^{-1}T^{-2}]^b[MT^{-1}L^{-1}]^c$$

Equating the exponents for the three-dimensional quantities, we have:

$$\text{For } M: \qquad 0 = a + b + c$$

$$\text{For } L: \qquad 1 = -3a - b - c$$

$$\text{For } T: \qquad -1 = -2b - c$$

Solving these three equations, it follows that

$$a = -\tfrac{1}{2} \qquad b = \tfrac{1}{2} \qquad c = 0$$

Hence from Eqn. 10.31

$$v = K\sqrt{\frac{p}{\rho}}$$

Thus the velocity of sound does not depend on the viscosity of the gas through which the wave is passing. By measuring the velocity of sound in a gas of known pressure and density it is possible to determine the constant K.

It can readily be appreciated that if some physical quantity is a function of four or more other physical quantities no unique solution can be obtained as was done in the above problem. It is usual then to form independent dimensional groups among the quantities. If a problem can be formulated in terms of p secondary quantities such as density, viscosity, force, pressure, etc., and if these p quantities can be expressed in terms of q primary quantities which are frequently mass, length, and time, then $p - q$ independent dimensionless groups can be formed.

10.13 The Use of Models and Dimensional Analysis

Dimensional analysis has provided a powerful tool for investigating difficult problems dealing with airplanes, ships, etc. It is used to predict the forces on full-scale airplanes and ships from measurements of the corresponding forces on suitable models. Suppose that F is the resistance force exerted on a body having some characteristic length l when it is moved through a fluid of density ρ and viscosity η with a velocity v relative to the fluid. The length l might be the chord of a wing or some length associated with the beam of a ship.

The force F is a function of the above variables and may be stated as follows:

$$F = f(l, \rho, \eta, v)$$

The force F may then be written

$$F = Kl^a\rho^b\eta^c v^d \tag{10.32}$$

where K is a dimensionless constant and the exponents are dimensionless constants which have to be determined. By inserting the dimensions of the various quantities, we have

$$[MLT^{-2}] = [L]^a[ML^{-3}]^b[ML^{-1}T^{-1}]^c[LT^{-1}]^d$$

Since the exponents of the three primary dimensional quantities must each be equal to zero, it follows that:

$$\text{For } M: \qquad 1 = b + c$$

$$\text{For } L: \qquad 1 = a - 3b - c + d$$

$$\text{For } T: \qquad -2 = -c - d$$

These equations may be solved to give

$$a = 2 - c$$

$$b = 1 - c$$

$$d = 2 - c$$

Equation 10.32 can then be written

$$F = l^{2-c}\rho^{1-c}\eta^c v^{2-c}$$

$$= (l^2\rho v^2)\left(\frac{\eta}{l\rho v}\right)^c$$

or

$$\frac{F}{l^2\rho v^2} = f\left(\frac{\eta}{l\rho v}\right) \tag{10.33}$$

Since in Eqn. 9.27 we have introduced Reynolds' number R as $l\rho v/\eta$, we shall again use it here. Thus Eqn. 10.33 may equally well be written

$$\frac{F}{l^2v^2\rho} = f_1\left(\frac{l\rho v}{\eta}\right) \tag{10.34}$$

Each side of Eqn. 10.34 represents a dimensionless group. Other equations having dimensionless groups on each side could have been obtained from the three equations between a, b, c, and d. From

Eqn. 10.34 it follows that the force exerted on a body in its motion through a viscous fluid can be represented by a curve showing the quantity $F/l^2v^2\rho$ plotted as ordinate against Reynolds' number $l\rho v/\eta$ as abscissa. Such a curve is called the characteristic curve and depends on the shape of the body but is independent of the size of the body. This curve defines the function f_1 in Eqn. 10.34. Once the curve is constructed for the model it is equally valid for the full-scale airplane or ship.

Suppose that conditions can be arranged so that $l\rho v/\eta$ is the same for an airplane wing as for its model. Then if F_1 is the measured force on the wing model having a chord length l_1, wind velocity v_1, and density ρ_1, the force F on the full-scale wing having a chord length l, a wind velocity v, and an air density ρ is given by Eqn. 10.34 as

$$\frac{F}{l^2v^2\rho} = \frac{F_1}{l_1^2v_1^2\rho_1}$$

or

$$F = F_1\frac{l^2v^2\rho}{l_1^2v_1^2\rho_1}$$

Now, if $l\rho v/\eta$ is the same for the wing and its model, then, from the above notation in which the subscript refers to the model,

$$\frac{l\rho v}{\eta} = \frac{l_1\rho_1v_1}{\eta_1}$$

If the model is placed in a wind tunnel where

$$\eta = \eta_1 \qquad \text{and} \qquad \rho = \rho_1$$

then in order for Reynolds' number to have the same value it is necessary that

$$v_1 = \frac{vl}{l_1} \tag{10.35}$$

Now, since the ratio l/l_1 is usually very large, it follows from Eqn. 10.35 that the velocity of the air in the wind tunnel must be much larger than the velocity of the plane in flight. These conditions cannot be carried out in practice. What is frequently done is to construct a wind tunnel in which the density of the air ρ_1 is larger than the density ρ at atmospheric pressure. By this means fairly large values of Reynolds' number can be obtained without excessive velocities. Thus results applicable to large planes can be obtained with relatively small models in pressure wind tunnels.

10.14 A Final Word

In the preceding pages an elementary picture of classical mechanics has been given. There are many more complicated methods of analysis in classical mechanics than have been presented in this book, but these must be left to more advanced treatises. What has been covered here should give you a good preparation for further study. Modern physics as it has been developed in the present century forms such an important part of the whole of physics that you may wonder why you should spend so much time on classical physics The fact is that, although modern physics has made some breaks with classical physics, nevertheless, on the whole, there is a continuity of ideas running through all of physics. The laws of conservation of momentum and energy are just as valid today as they were before the turn of the century. Modern physics cannot really be understood without a sound knowledge of classical physics.

PROBLEMS

1. Show that the one-dimensional wave equation, Eqn. 10.3, is satisfied by the following functions:

a. $y = A(x - vt)$

b. $y = A(x - vt)^2$

c. $y = A(\sqrt{x + vt})$

d. $y = A \log_e(x + vt)$

where $v = \sqrt{F/\mu}$ and A is a constant.

2. Explain why the functions given in problem 1 are not useful in discussing wave motion.

3. Show that F/μ has the dimensions of the square of the velocity, using both the absolute and gravitational systems for the force F and mass per unit length μ

4. Show that Eqn. 10.21 for a standing wave is a solution of the wave equation, Eqn. 10.3.

5. Show that the energy per unit volume for a plane longitudinal wave of amplitude A and angular velocity ω in a homogeneous rod of density ρ is $(A^2\omega^2\rho)/2$.

6. Show that the velocity of sound in a gas is given by Eqn. 10.15, or $v = \sqrt{\gamma P/\rho}$.

7. *a.* Find the velocity of sound in air at 1 atm of pressure and 0°C if $\gamma = 1.41$.

b. Find the velocity at 0°C and 2 atm of pressure.

c. Find the velocity at 100°C and 1 atm of pressure.

8. Prove that, in general, the velocity v of waves in a medium is given by

$$v = \sqrt{\frac{\text{Modulus of elasticity}}{\text{Density of medium}}}$$

9. An explosion creates a sound wave which travels uniformly out in all directions from a point. Explain why the displacement ξ of the air at any distance r from the center may be written

$$\xi = \frac{A}{r} \sin k(r - vt)$$

To do this consider the energy per second crossing a spherical surface of radius r. Show that the intensity of the wave varies as the inverse square of the distance from the source.

10. A meter stick whose mass is 120 gm is clamped horizontally so that 80 cm projects over the edge of a table. When a 120-gm load is hung on one end, a further deflection of 2 cm occurs. The width of the stick is 3 cm, and its thickness is 0.5 cm. The stick is placed so that the width is originally horizontal. Find:

a. Young's modulus for the rod.

b. The velocity of longitudinal waves in the rod.

11. A note whose frequency is 250 vibrations per second has an intensity of 1 microwatt per square meter. Find the amplitude of the vibrations in the air created by this audible sound. The velocity of sound in air is about 331 meters per second.

12. The displacement ξ in a plane sound wave is given by

$$\xi = A \sin \frac{2\pi}{\lambda}(x - vt)$$

Prove that for this wave the excess pressure in a gas above normal atmospheric pressure may be expressed as

$$p = -2\pi A \frac{\rho v^2}{\lambda} \cos \frac{2\pi}{\lambda}(x - vt)$$

Note: For a gas Eqn. 10.11 becomes

$$\frac{\rho \partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x}$$

13. Find the decrease in pressure below normal atmospheric pressure in a rarefaction, using the data and result of problem 11.

14. Prove that the pressure above normal atmospheric pressure in a standing wave is given by

$$p = -\frac{4\pi v^2 \rho A}{\lambda} \cos 2\pi \frac{t}{P} \cos 2\pi \frac{x}{\lambda}$$

15. Prove that in a standing wave a pressure node occurs where the velocities of the particles or molecules of air are at a maximum and conversely that a velocity node occurs where the pressure is at a maximum.

16. Show that at the closed end of an organ pipe a compression is reflected as a compression, or that the reflected pressure wave is in phase with the incident pressure wave, and that the total excess pressure at the closed end is twice the excess pressure in each of the waves. (At a closed end the resultant velocity of the particles is zero.)

17. Show that at the open end of an organ pipe a compression is reflected as a rarefaction. (At the open end there is a pressure node and a velocity maximum or loop.)

18. The phase velocity of waves in deep water is proportional to the square root of their wave length. Show that the group velocity is equal to one-half the phase velocity.

19. About 1924 a French physicist, L. de Broglie, announced a new theory according to which a particle of mass m moving with a velocity V has associated with it waves whose wave length is $\lambda = h/mV$. In this theory h is a constant called Planck's constant. The phase velocity of these waves was given as $v =$

c^2/V, where c is the velocity of light in a vacuum. This phase velocity is greater than c, the velocity of light, and this is contrary to the restricted theory of relativity. Prove that the group velocity of these waves is equal to the velocity V of the mass m. Use the equation of Chapter 1 showing that the mass m of a particle moving with a velocity V is given in terms of the rest mass of the particle as

$$m = \frac{m_0}{\sqrt{1 - (V^2/c^2)}}$$

20. About 1885 Michelson measured the velocity of light in air and in carbon disulphide and found their ratio to be 1.758 He also measured the refractive index n of carbon disulphide and found it to be 1.635. According to the wave theory of light the refractive index for a medium is the ratio of the phase velocity of a wave in air to that in the medium. Thus for carbon disulphide $n = c/v = 1.635$, where c is the velocity of light in air or a vacuum. Direct measurement of the velocity of light gives the group velocity so that for carbon disulphide $c/u = 1.758$. Note that the group velocity is

$$u = v - \lambda_m \frac{dv}{d\lambda_m}$$

where λ_m is the wave length of the light in the medium and $\lambda_m = \lambda/n$. Show that

$$\frac{c}{u} = n - \lambda \frac{dn}{d\lambda}$$

and that this theory is in agreement with Michelson's results for light of wave length $\lambda = 5.8 \times 10^{-5}$ cm at which the mean rate of change of refractive index with wave length is -2.1×10^3 per cm.

21. Prove the result given in Eqn. 10.25:

$$A_m = \frac{1}{l} \int_0^l y_0 \sin \frac{\pi m x}{l} \, dx$$

Also prove that in the Fourier coefficient

$$B_m = \frac{1}{\pi v m} \int_0^l y_0 \sin \frac{\pi m x}{l} \, dx$$

Where $m = 1, 2, 3, 4 \cdots$ and $\dot{y}_0$ is the initial velocity of the cord.

22. A cord of length l is rigidly attached at both ends and is plucked to a height h at a point $\frac{1}{3}$ from one end. If the velocity of transverse waves on the cord is v, show that in the subsequent vibration the third, sixth, and ninth harmonics are absent and that the displacement y at any distance x along the string at time t is given by

$$y = \frac{3^{5/2} h}{2\pi^2} \left(\sin \frac{\pi x}{l} \cos \frac{\pi v t}{l} + \frac{1}{4} \sin \frac{2\pi x}{l} \cos \frac{2\pi v t}{l} - \frac{1}{16} \sin \frac{4\pi x}{l} \cos \frac{4\pi v t}{l} \cdots \right)$$

23. The cord in question 22 has a mass of 0.01 gm per cm and is subjected to a tension of 4×10^6 dynes. If its length is 50 cm and it is plucked to an initial height of $h = 2$ cm, find the energy per unit length of the cord of the fundamental and the first two harmonics.

24. A cord of length l and rigidly attached at both ends is struck at its center.

The velocity of the transverse waves in the string is v. If at time $t = 0$ the displacement y_0 of the cord is zero but the initial velocity of its center is V and this changes linearly to zero at both ends, show that the Fourier coefficients A_m are all zero and that the even harmonics, that is, those with a node at the center, are absent. Show that the displacement y at any point x at time t is given by

$$y = \frac{8Vl}{\pi^2 v}\left(\sin\frac{\pi x}{l}\sin\frac{\pi vt}{l} - \frac{1}{27}\sin\frac{3\pi x}{l}\sin\frac{3\pi vt}{l} + \frac{1}{125}\sin\frac{5\pi x}{l}\sin\frac{5\pi vt}{l} - \frac{1}{343}\sin\frac{7\pi x}{l}\sin\frac{7\pi vt}{l} \cdots\right)$$

25. The cord in problem 24 is struck at a point one-quarter of the length from one end. At time $t = 0$ the displacement of the cord is zero while the velocity changes linearly from V at $l/4$ to zero at both $x = 0$ and $x = l/2$ and is zero from $l/2$ to l. Show that the fourth and eighth harmonics are absent, that is, those having nodes at $x = l/4$. Show that the displacement y at any point x and time t is given by

$$y = \frac{8Vl}{\pi^2 v}\left[(\sqrt{2} - 1)\sin\frac{\pi x}{l}\sin\frac{\pi vt}{l} + \frac{1}{4}\sin\frac{2\pi x}{l}\sin\frac{2\pi vt}{l} + \frac{1+\sqrt{2}}{27}\sin\frac{3\pi x}{l}\sin\frac{3\pi vt}{l} \cdots\right]$$

26. From dimensional analysis find the velocity of sound in a solid, assuming that the velocity v depends on the density ρ, Young's modulus E, and the acceleration of gravity g.

27. Waves on the surface of a liquid may be divided into two classes: those due to surface tension and those due to gravity. Surface tension waves are the short wave-length ripples, and their velocity is a function of the surface tension of the liquid, their wave length λ, and the density ρ of the liquid. Gravity waves may be divided into two classes, depending on the relative size of the wave length and the depth of the liquid. If the wave length is short compared to the depth, the velocity of these deep sea waves depends on their wave length λ and the acceleration due to gravity g. On the other hand, if the wave length is large compared to the depth, the velocity depends on the depth h and the acceleration g due to gravity. By dimensional analysis find the velocity of the waves in each of the three cases.

28. By means of dimensional analysis and the data given below find the complete equation for the velocity of sound in a solid rod. The velocity v depends on the weight per unit volume ρ_w, Young's modulus E, and the acceleration due to gravity g. The following data was found for a rod of steel: $v = 1.65 \times 10^4$ ft per sec; $\rho_w = 490$ lbw per ft^3; $E = 4.16 \times 10^9$ lbw per ft^2. If the density of aluminum is 168 lbw per ft^3 and Young's modulus for this substance is 1.30×10^9 lbw per ft^2, find the velocity of sound in an aluminum rod.

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